Quantum Criticality and Black Holes

Talk online: sachdev.physics.harvard.edu

Particle theorists

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1. CFT3s in condensed matter physics and string theory
   Superfluid-insulator transition, magnetic ordering transitions, graphene ....

2. Quantum-critical transport
   Collisionless-to-hydrodynamic crossover of CFT3s

3. Black Hole Thermodynamics
   Connections to quantum criticality

4. Generalized magnetohydrodynamics
   Quantum criticality and dyonic black holes

5. Experiments
   Graphene and the cuprate superconductors

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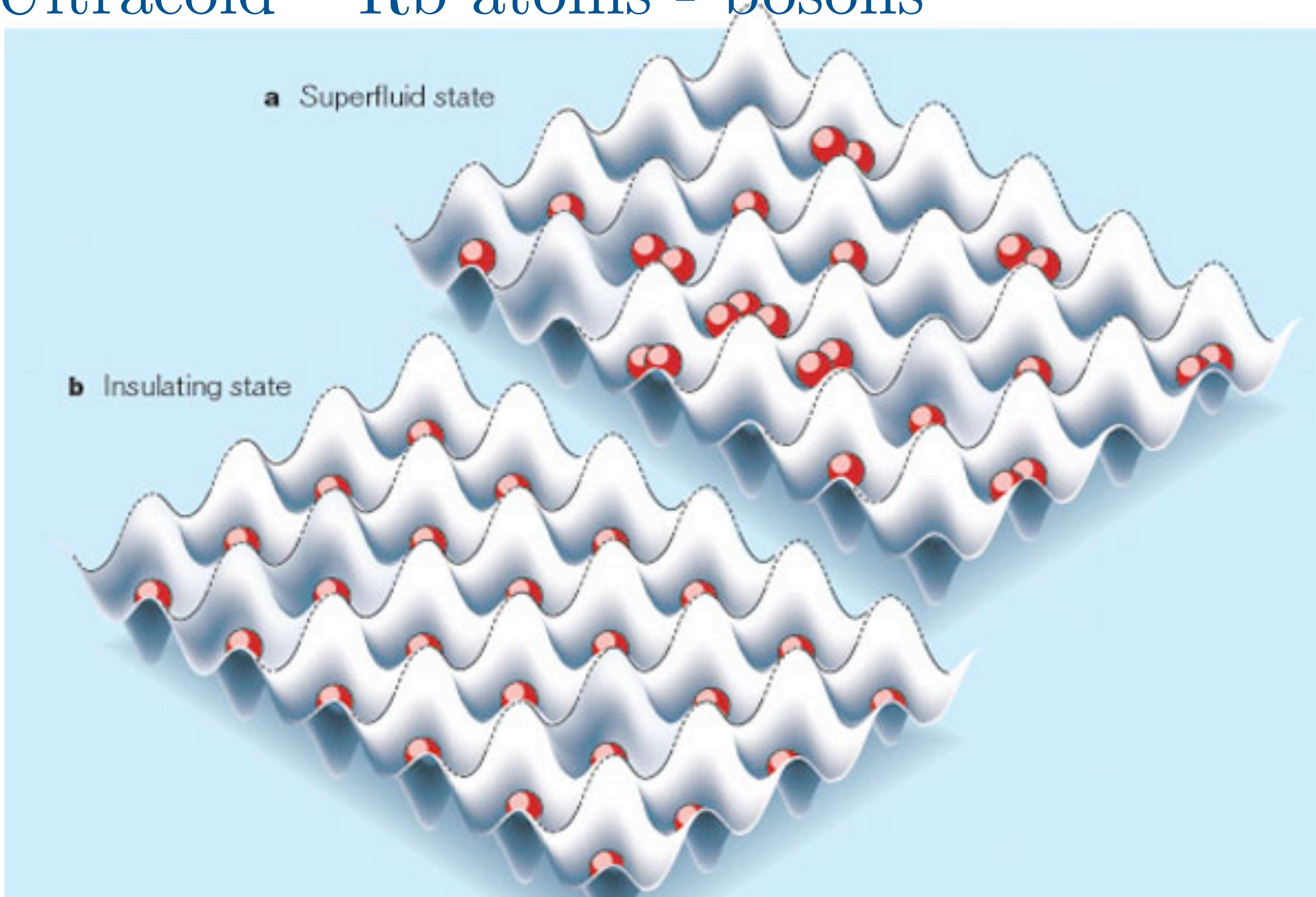
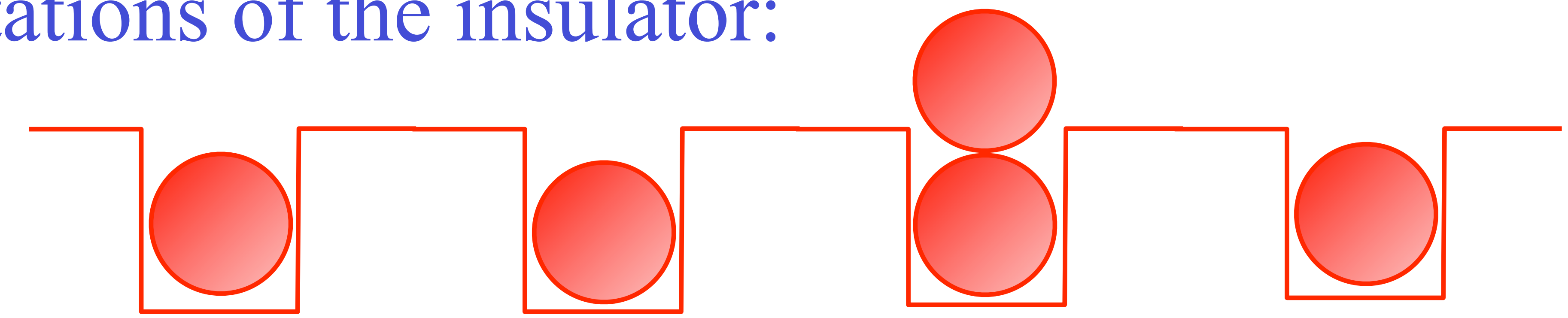
Ultracold ^{87}Rb atoms - bosons



M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch, *Nature* **415**, 39 (2002).

The insulator:

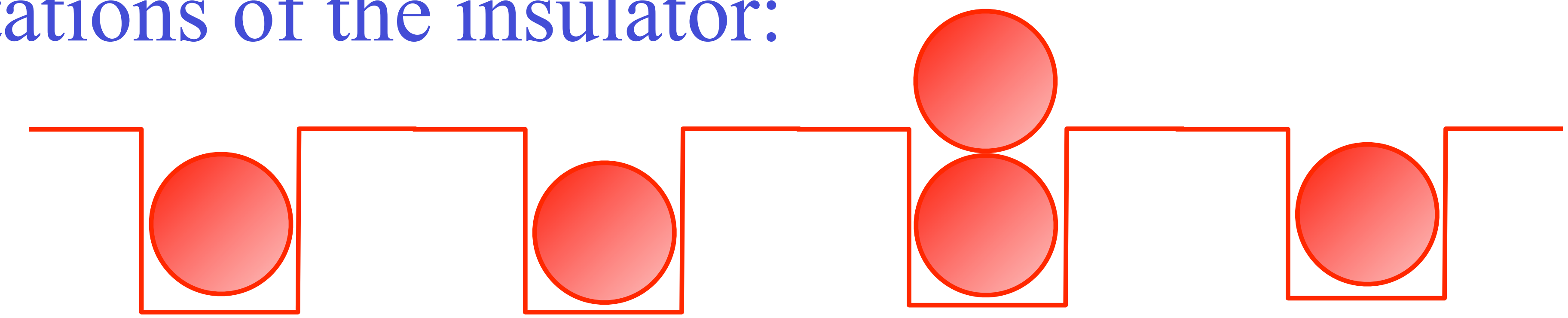
Excitations of the insulator:



Particles $\sim \psi^{\dagger}$

Holes $\sim \psi$

Excitations of the insulator:



Particles $\sim \psi^{\dagger}$

Holes $\sim \psi$

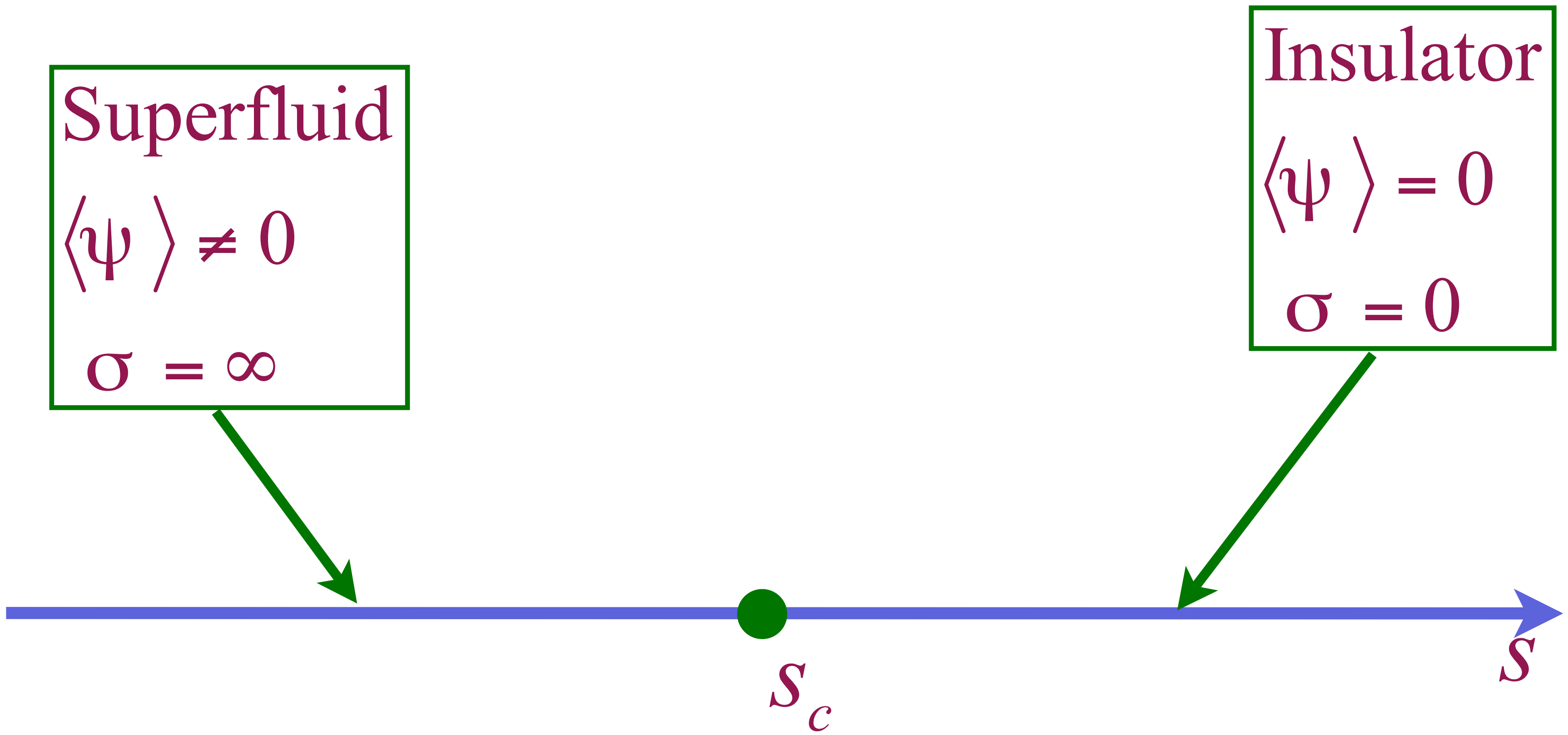
Density of particles = density of holes $\Rightarrow$

"relativistic" field theory for ψ:

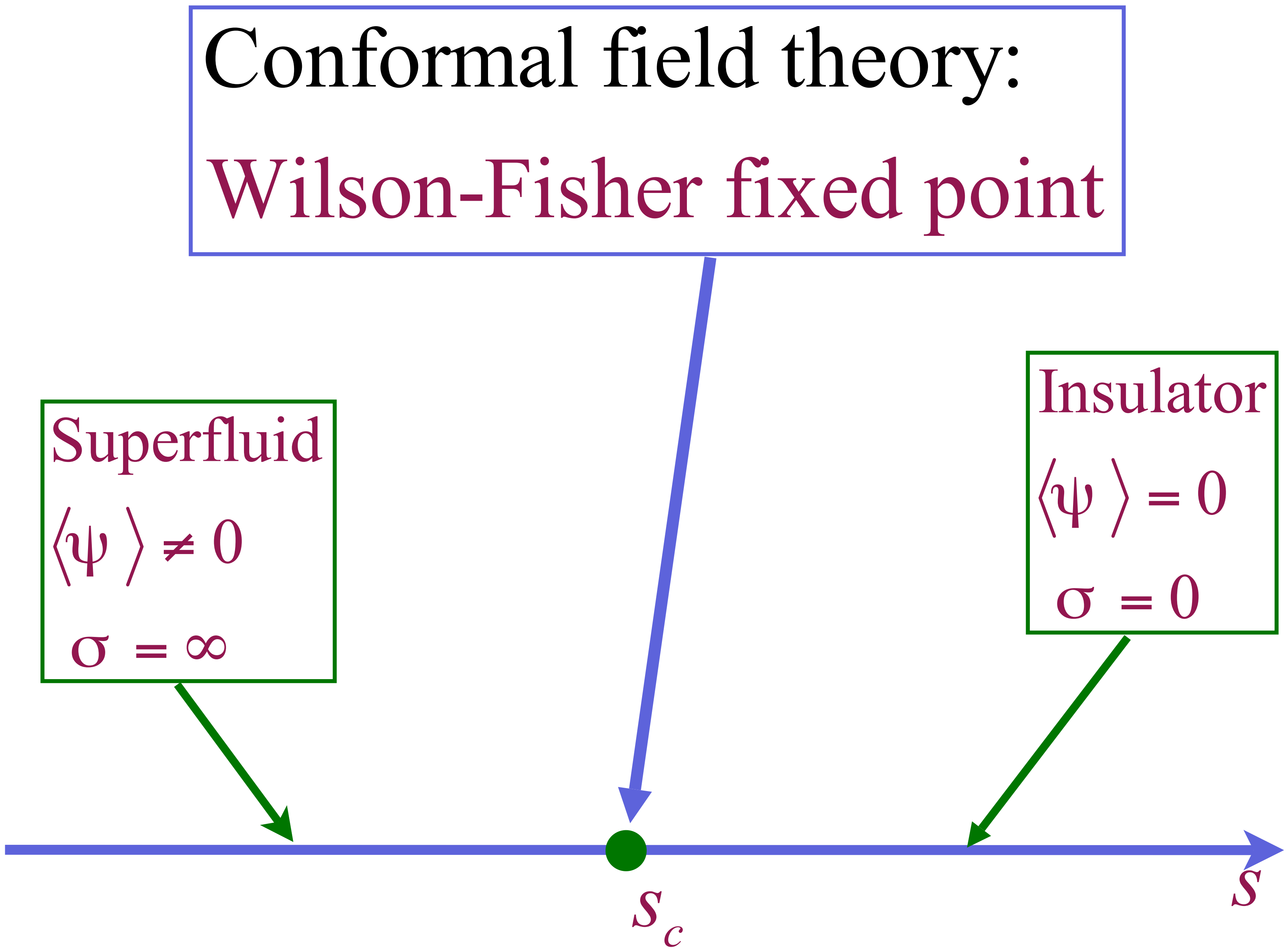
$$\mathcal{S} = \int d^2 r d\tau \left[|\partial_\tau \psi|^2 + c^2 |\vec{\nabla}\psi|^2 + s|\psi|^2 + \frac{u}{2}|\psi|^4 \right]$$

Insulator $\Leftrightarrow$ $\langle \psi \rangle = 0$

Superfluid $\Leftrightarrow$ $\langle \psi \rangle \neq 0$

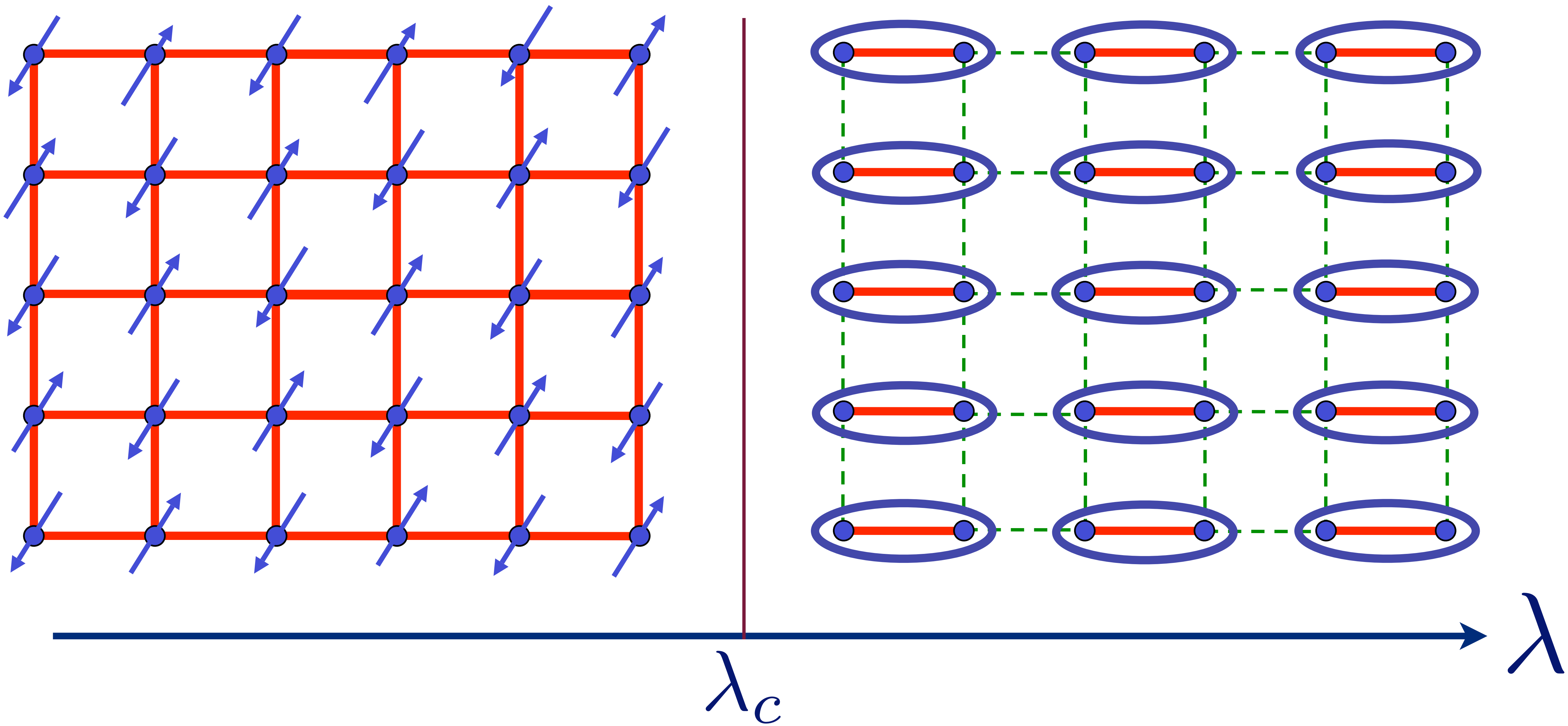


$$\mathcal{S} = \int d^2 r d\tau \left[|\partial_\tau \psi|^2 + c^2 |\vec{\nabla}\psi|^2 + s|\psi|^2 + \frac{u}{2}|\psi|^4 \right]$$



$$\mathcal{S} = \int d^2 r d\tau \left[|\partial_\tau \psi|^2 + c^2 |\vec{\nabla} \psi|^2 + s|\psi|^2 + \frac{u}{2} |\psi|^4 \right]$$

Coupled dimer antiferromagnet



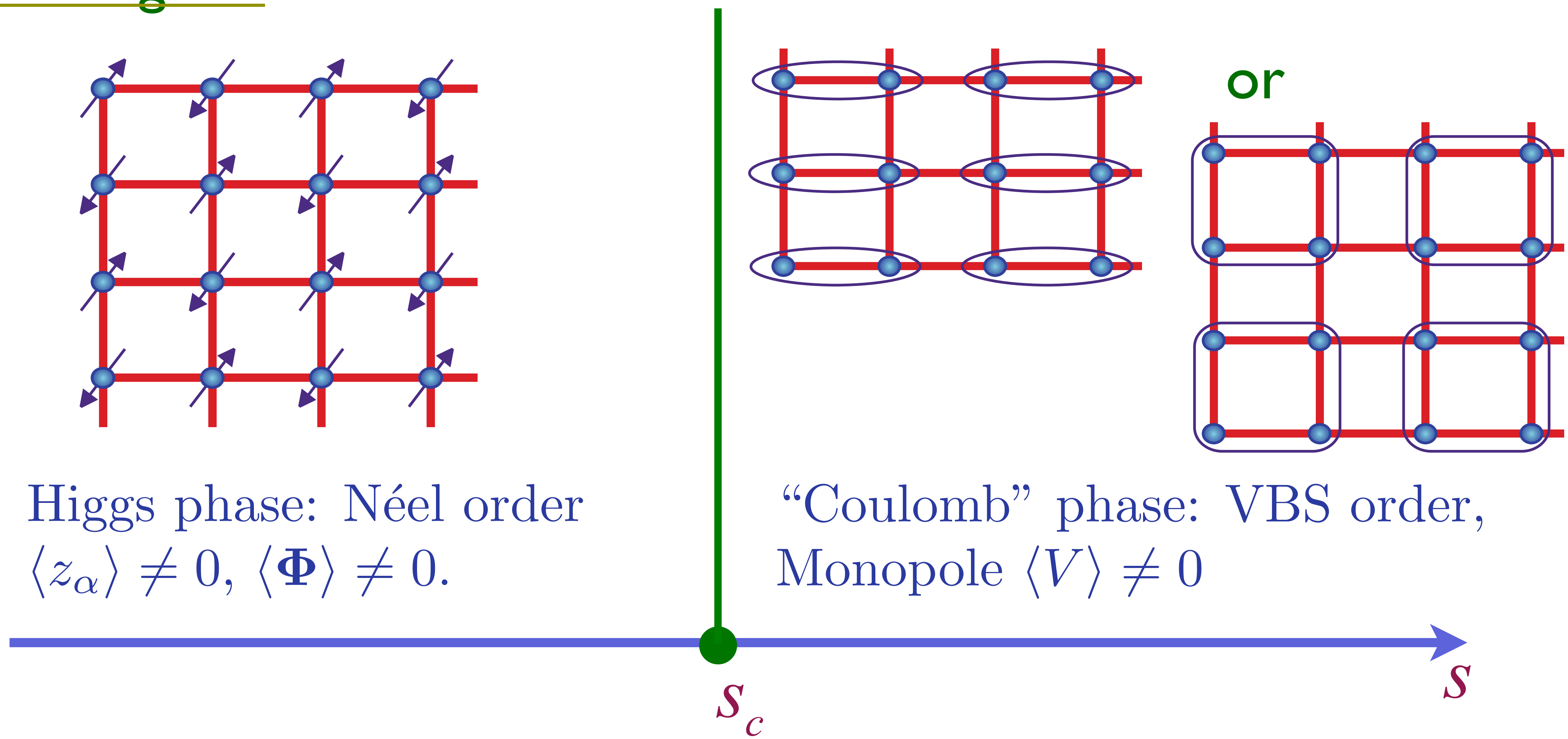
O(3) order parameter $\boldsymbol{\Phi} = (-1)^i \boldsymbol{S}_i$

$$\mathcal{S} = \int d^2r d\tau \left[(\partial_\tau \boldsymbol{\Phi})^2 + c^2 (\vec{\nabla} \boldsymbol{\Phi})^2 + s \boldsymbol{\Phi}^2 + u \left(\boldsymbol{\Phi}^2 \right)^2 \right]$$

S=1/2 Heisenberg antiferromagnets on the square lattice

$$H = J\sum_{\langle ij\rangle}\mathbf{S}_i\cdot\mathbf{S}_j - Q\sum_{\langle ijkl\rangle}\left(\mathbf{S}_i\cdot\mathbf{S}_j-\tfrac{1}{4}\right)\left(\mathbf{S}_k\cdot\mathbf{S}_l-\tfrac{1}{4}\right)$$

Phase diagram



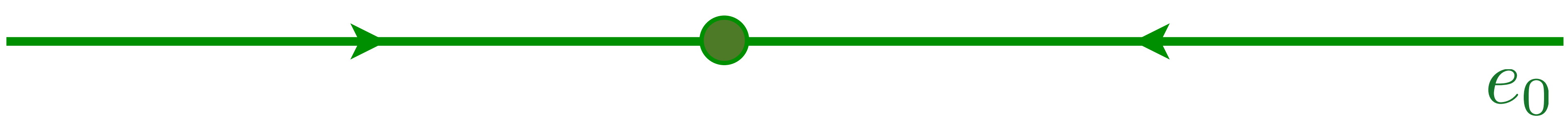
$$\mathcal{S}_z = \int d^2r d\tau\left[|(\partial_\mu - iA_\mu)z_\alpha|^2 + s|z_\alpha|^2 + u(|z_\alpha|^2)^2 + \frac{1}{2e_0^2}(\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda)^2\right]$$

Similar phase diagram for $\mathcal{N}=4$ SQED with 2 matter hypermultiplets

S=1/2 Heisenberg antiferromagnets on the square lattice

Algebraic spin liquids

$$\mathcal{S} = \int d^2 r d\tau \left[\overline{\psi}_a \gamma_\mu (\partial_\mu - i A_\mu) \psi_a + \frac{1}{2e_0^2} (\epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda)^2 \right]$$

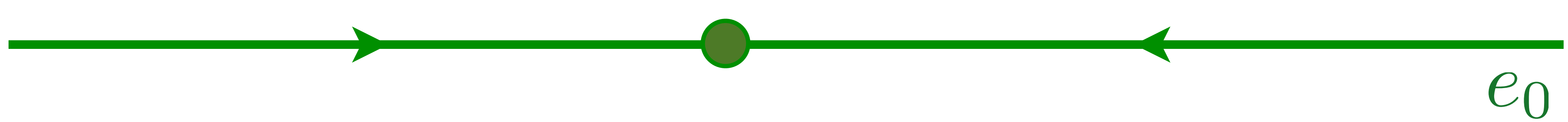


RG flow to an attractive fixed point

S=1/2 Heisenberg antiferromagnets on the square lattice

Algebraic charge liquids

$$\mathcal{S} = \int d^2 r d\tau \Big[|(\partial_\mu - i A_\mu) z_\alpha|^2 + s|z_\alpha|^2 + u(|z_\alpha|^2)^2 + \overline{\psi}_a \gamma_\mu (\partial_\mu - i A_\mu)\psi_a + \frac{1}{2e_0^2}(\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda)^2 \Big]$$

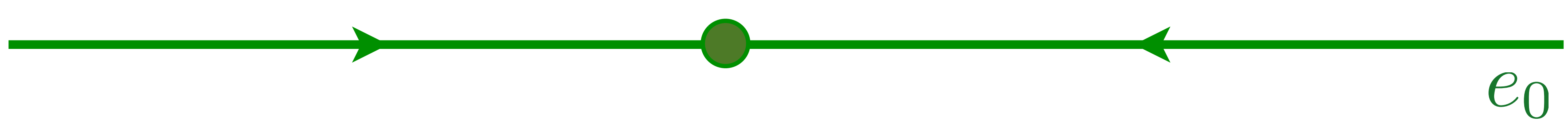


Loss of Néel order in a d-wave superconductor

R. Kaul et al., Nature Physics 4, 28, (2008)

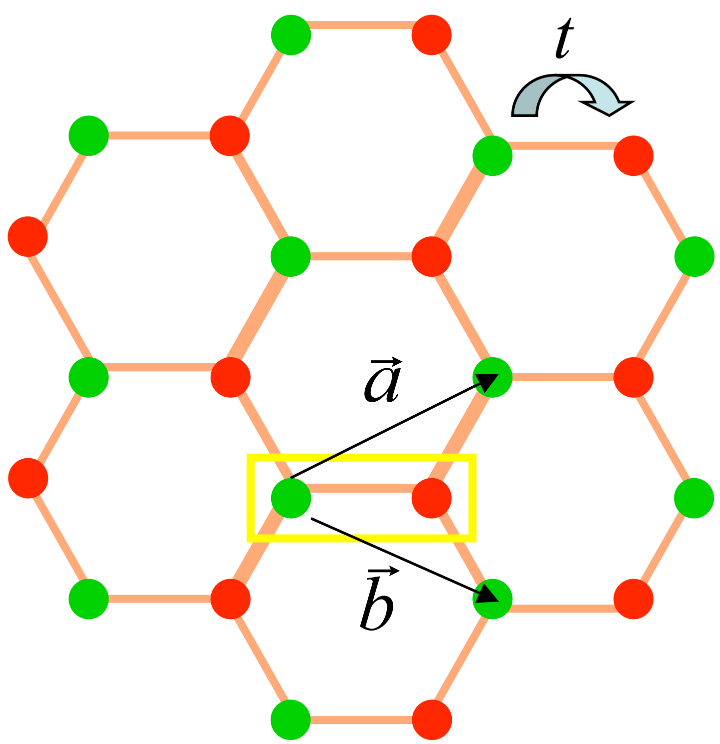
SU(N) gauge theory with $\mathcal{N} = 8$ supersymmetry

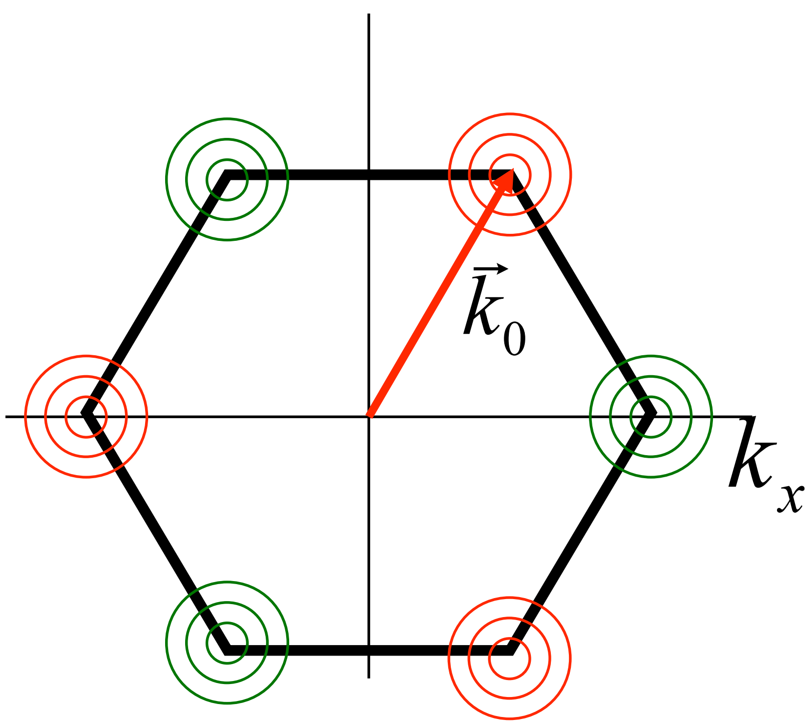
Theory of a Yang-Mills gauge field A_μ coupled to relativistic scalars and fermions. Characterized by a single gauge coupling constant e_0.

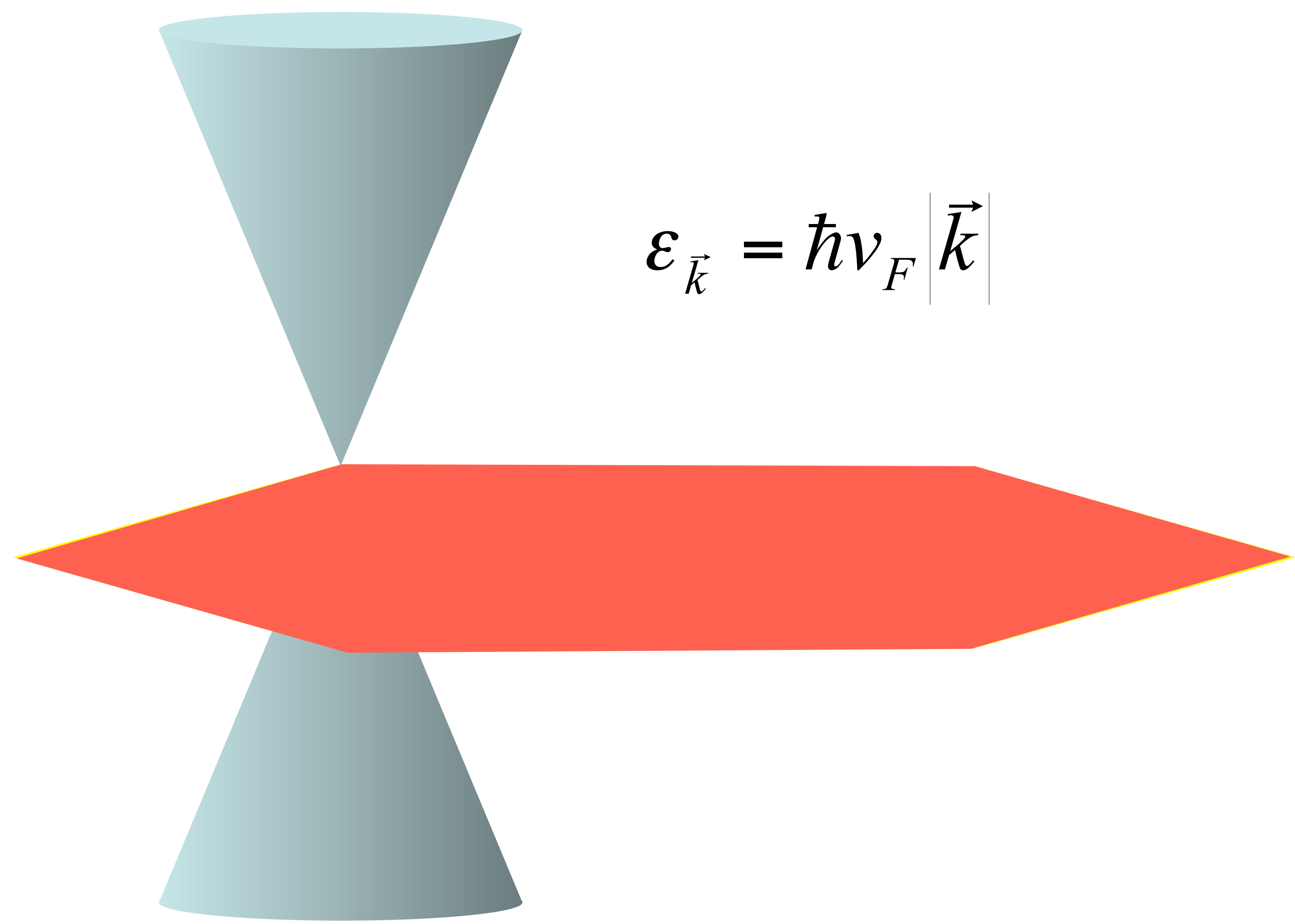


RG flow to an attractive fixed point

Graphene







$$\varepsilon_{\vec{k}} = \hbar v_F \left|\vec{k}\right|$$

Graphene

Low energy theory has 4 two-component Dirac fermions, ψ_α, $\alpha = 1 \ldots 4$, interacting with a $1/r$ Coulomb interaction

$$\begin{aligned} \mathcal{S} &= \int d^2 r d\tau \psi_\alpha^\dagger \left(\partial_\tau - i v_F \vec{\sigma} \cdot \vec{\nabla} \right) \psi_\alpha \\ &\quad + \frac{e^2}{2} \int d^2 r d^2 r' d\tau \psi_\alpha^\dagger \psi_\alpha(r) \frac{1}{|r - r'|} \psi_\beta^\dagger \psi_\beta(r') \end{aligned}$$

Dimensionless "fine-structure" constant $\alpha = e^2/(4\hbar v_F)$.
RG flow of α:

$$\frac{d\alpha}{d\ell} = -\alpha^2 + \ldots$$

Behavior is similar to a CFT3 with $\alpha \sim 1/\ln(\textbf{scale})$.

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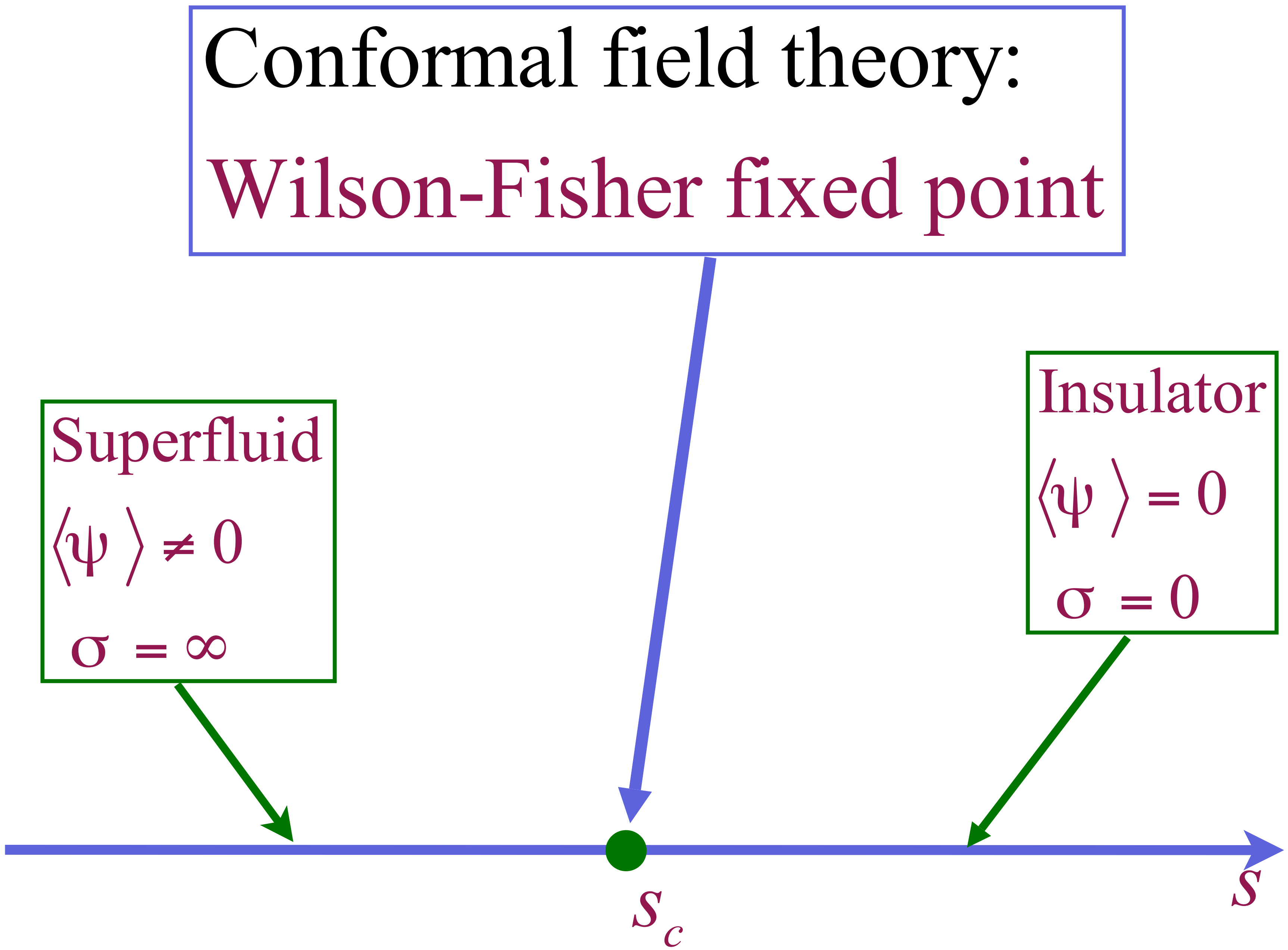
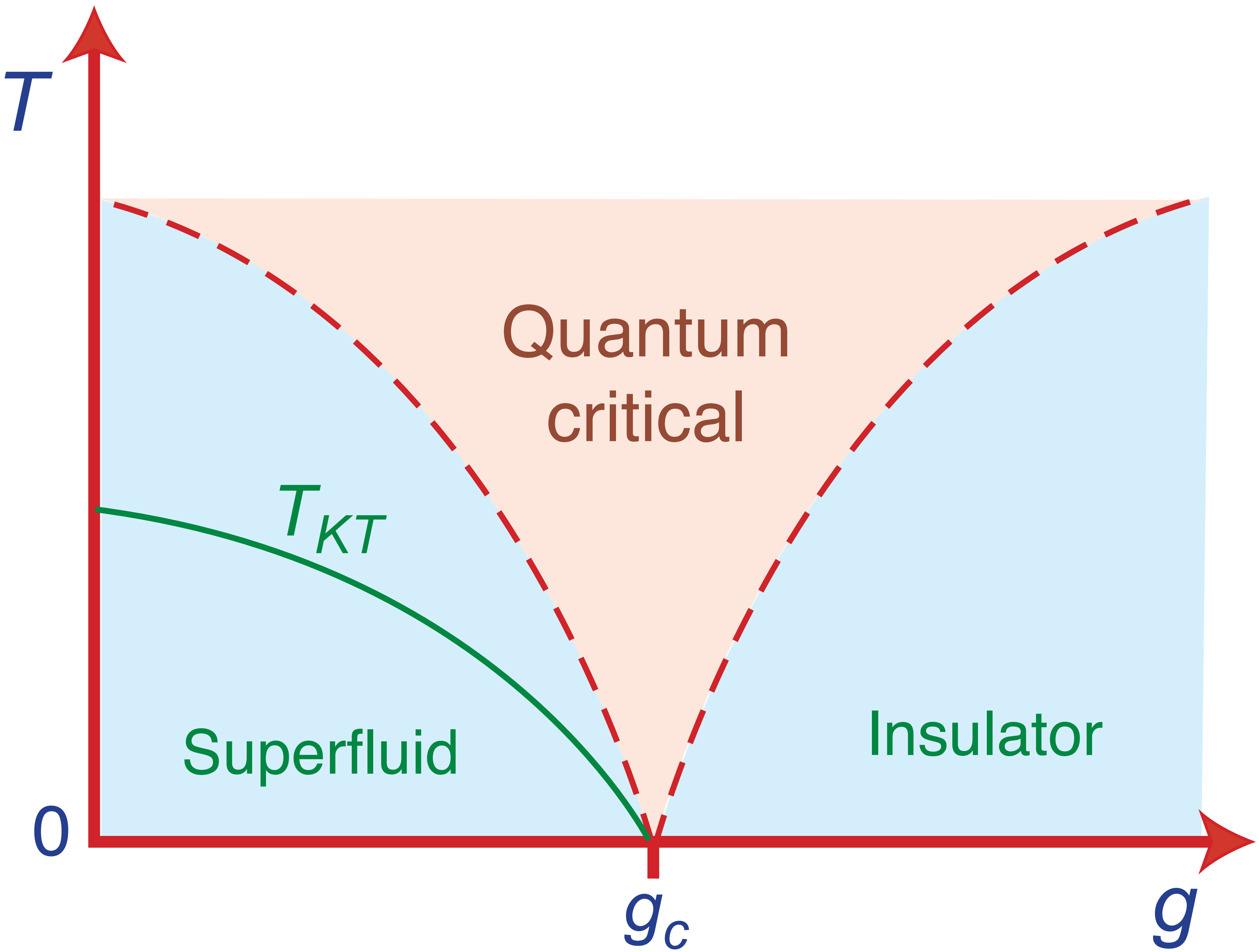
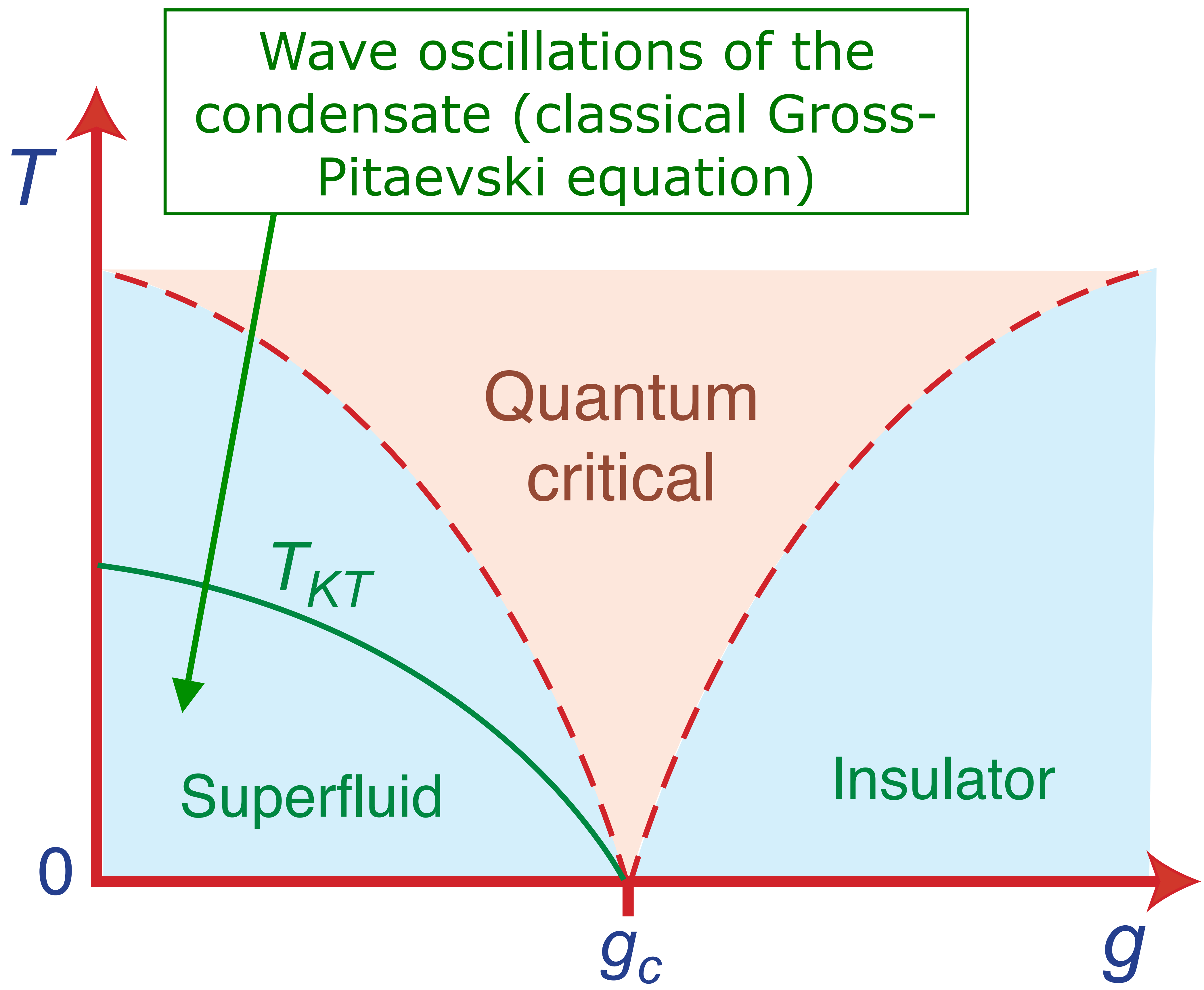
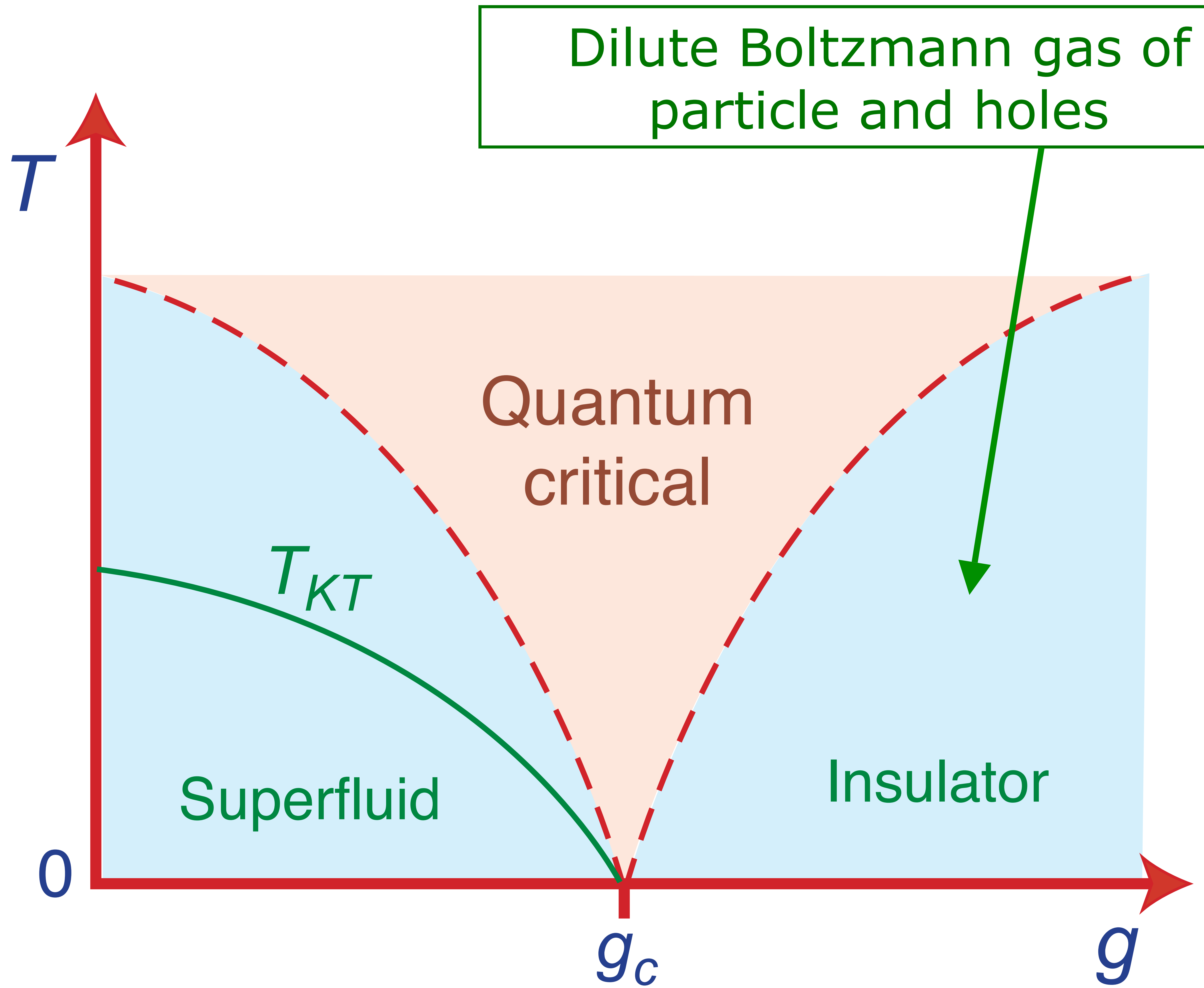


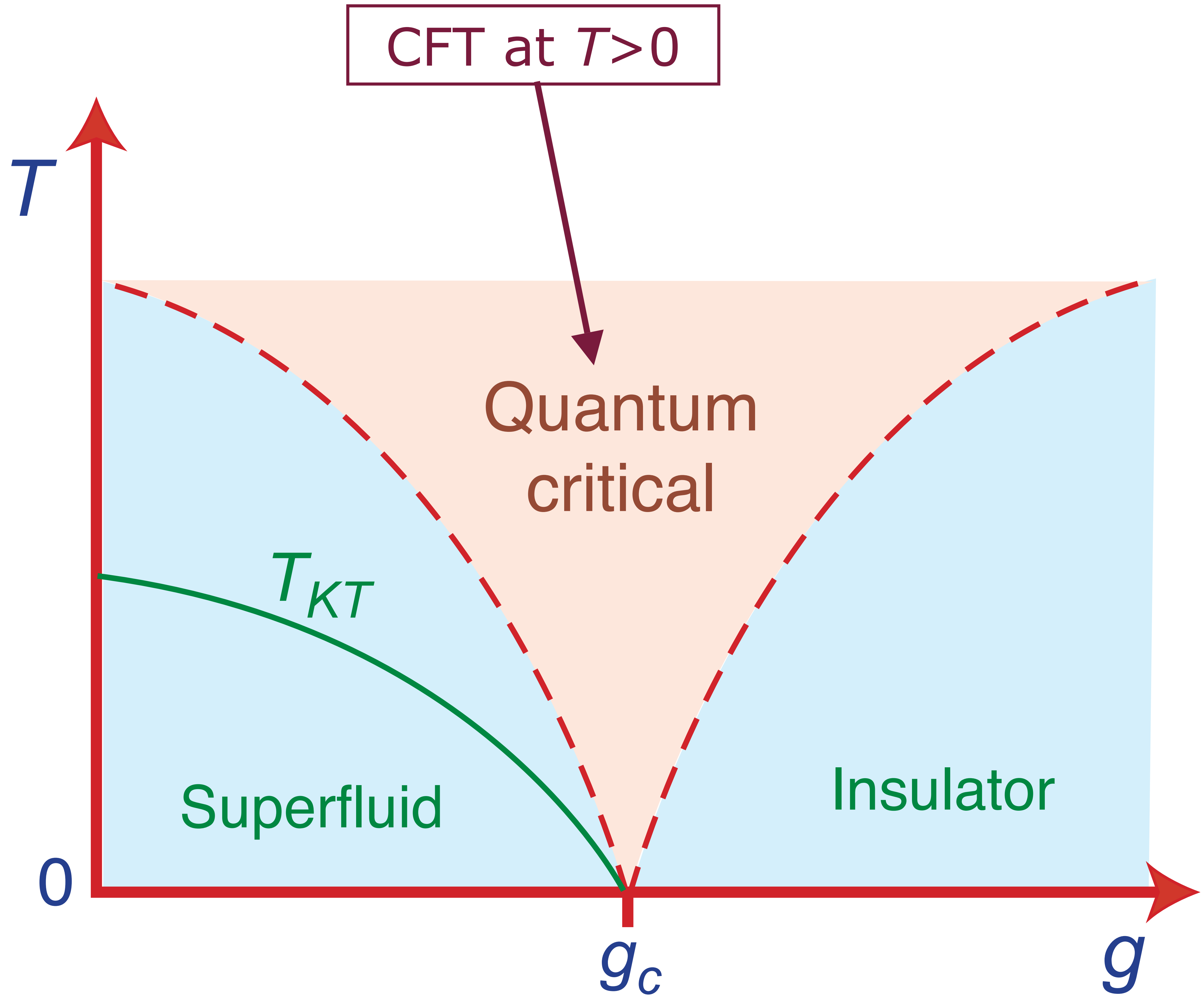
$$\mathcal{S} = \int d^2 r d\tau \left[|\partial_\tau \psi|^2 + c^2 |\vec{\nabla} \psi|^2 + s|\psi|^2 + \frac{u}{2}|\psi|^4 \right]$$



T
Quantum
critical
T_KT
Superfluid
Insulator
0
g_c
g


Wave oscillations of the condensate (classical Gross-Pitaevski equation)
T
Quantum critical
T_KT
Superfluid
Insulator
0
g_c
g


Dilute Boltzmann gas of particle and holes
T
Quantum critical
T_{KT}
Superfluid
Insulator
0
g_c
g


CFT at T>0
T
Quantum
critical
T_KT
Superfluid
Insulator
0
g_c
g

Resistivity of Bi films

Conductivity σ

$$\sigma_{\text{Superconductor}}(T \to 0) = \infty$$
$$\sigma_{\text{Insulator}}(T \to 0) = 0$$
$$\sigma_{\text{Quantum critical point}}(T \to 0) \approx \frac{4e^2}{h}$$

D. B. Haviland, Y. Liu, and A. M. Goldman, *Phys. Rev. Lett.* **62**, 2180 (1989)

M. P. A. Fisher, *Phys. Rev. Lett.* **65**, 923 (1990)

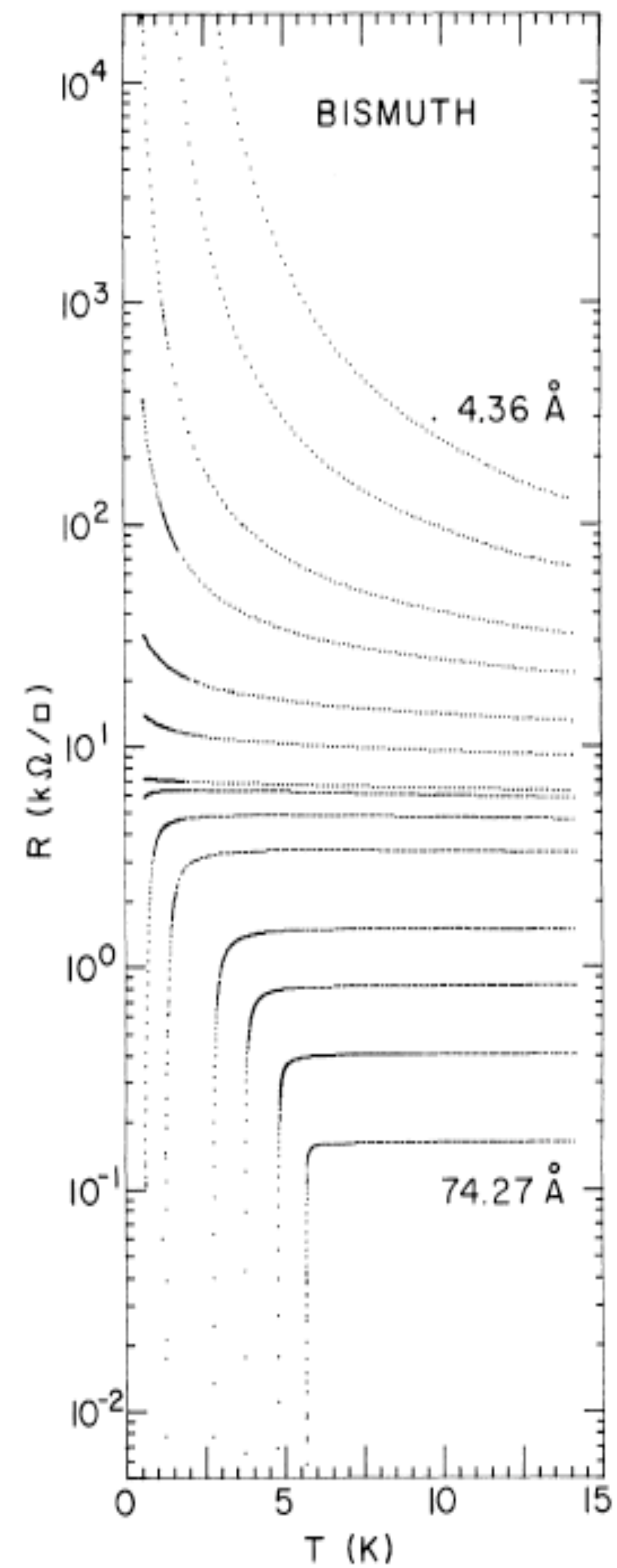


FIG. 1. Evolution of the temperature dependence of the sheet resistance $R(T)$ with thickness for a Bi film deposited onto Ge. Fewer than half of the traces actually acquired are shown. Film thicknesses shown range from 4.36 to 74.27 Å.

Density correlations in CFTs at $T > 0$

Two-point density correlator, $\chi(k,\omega)$

Kubo formula for conductivity $\sigma(\omega) = \lim_{k \to 0} \frac{-i\omega}{k^2} \chi(k,\omega)$

For *all* CFT2s, at all $\hbar\omega / k_B T$

$$\chi(k,\omega) \quad = \quad \frac{4e^2}{h} K \frac{vk^2}{v^2k^2 - \omega^2} \quad ; \quad \sigma(\omega) = \frac{4e^2}{h} \frac{Kv}{-i\omega}$$

where K is a universal number characterizing the CFT2 (the level number), and v is the velocity of "light".

Density correlations in CFTs at $T > 0$

Two-point density correlator, $\chi(k,\omega)$

Kubo formula for conductivity $\sigma(\omega) = \lim_{k\to 0} \frac{-i\omega}{k^2}\chi(k,\omega)$

For *all* CFT3s, at $\underline{\hbar\omega \gg k_B T}$

$$\chi(k,\omega) = \frac{4e^2}{h} K \frac{k^2}{\sqrt{v^2k^2 - \omega^2}} \quad ; \quad \sigma(\omega) = \frac{4e^2}{h} K$$

where K is a universal number characterizing the CFT3, and v is the velocity of "light".

Density correlations in CFTs at $T > 0$

Two-point density correlator, $\chi(k,\omega)$

Kubo formula for conductivity $\sigma(\omega) = \lim_{k\to 0} \frac{-i\omega}{k^2}\chi(k,\omega)$

However, for *all* CFT3s, at $\underline{\hbar\omega \ll k_B T}$, we have the Einstein relation

$$\chi(k,\omega) = 4e^2\chi_c \frac{Dk^2}{Dk^2 - i\omega} \quad ; \quad \sigma(\omega) = 4e^2 D\chi_c = \frac{4e^2}{h}\Theta_1\Theta_2$$

where the **compressibility**, χ_c, and the **diffusion constant** D obey

$$\chi = \frac{k_B T}{(hv)^2}\Theta_1 \quad ; \quad D = \frac{hv^2}{k_B T}\Theta_2$$

with Θ_1 and Θ_2 universal numbers characteristic of the CFT3

K. Damle and S. Sachdev, *Phys. Rev. B* **56**, 8714 (1997).

Density correlations in CFTs at $T > 0$

In CFT3s collisions are "phase" randomizing, and lead to relaxation to local thermodynamic equilibrium. So there is a crossover from collisionless behavior for $\hbar\omega \gg k_B T$, to hydrodynamic behavior for $\hbar\omega \ll k_B T$.

$$\sigma(\omega) = \begin{cases} \dfrac{4e^2}{h} K & , \quad \hbar\omega \gg k_B T \\[2ex] \dfrac{4e^2}{h} \Theta_1 \Theta_2 & , \quad \hbar\omega \ll k_B T \end{cases}$$

and in general we expect $K \neq \Theta_1 \Theta_2$ (verified for Wilson-Fisher fixed point).

K. Damle and S. Sachdev, *Phys. Rev. B* **56**, 8714 (1997).

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Black Holes

Objects so massive that light is gravitationally bound to them.

Black Holes

Objects so massive that light is gravitationally bound to them.

The region inside the black hole horizon is causally disconnected from the rest of the universe.

$$\text{Horizon radius } R = \frac{2GM}{c^2}$$

Black Hole Thermodynamics

Bekenstein and Hawking discovered astonishing connections between the Einstein theory of black holes and the laws of thermodynamics

Entropy of a black hole $S = \dfrac{k_B A}{4\ell_P^2}$

where A is the area of the horizon, and

$\ell_P = \sqrt{\dfrac{G\hbar}{c^3}}$ is the Planck length.

The Second Law: $dA \geq 0$

Black Hole Thermodynamics

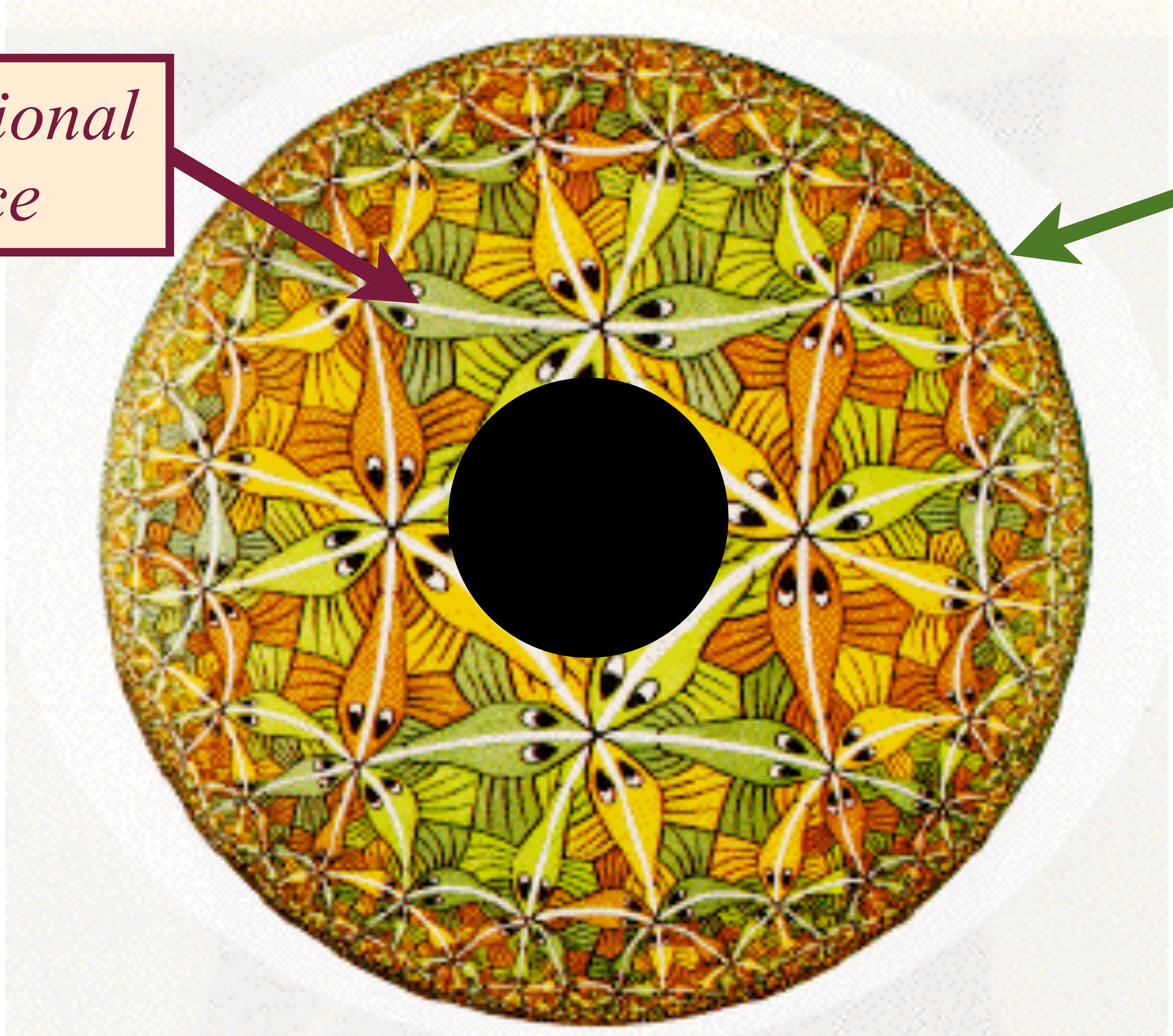
Bekenstein and Hawking discovered astonishing connections between the Einstein theory of black holes and the laws of thermodynamics

$$\text{Horizon temperature: } 4\pi k_B T = \frac{\hbar^2}{2M\ell_P^2}$$

AdS/CFT correspondence

The quantum theory of a black hole in a 3+1-dimensional negatively curved AdS universe is holographically represented by a CFT (the theory of a quantum critical point) in 2+1 dimensions

3+1 dimensional AdS space



A 2+1 dimensional system at its quantum critical point

Maldacena, Gubser, Klebanov, Polyakov, Witten

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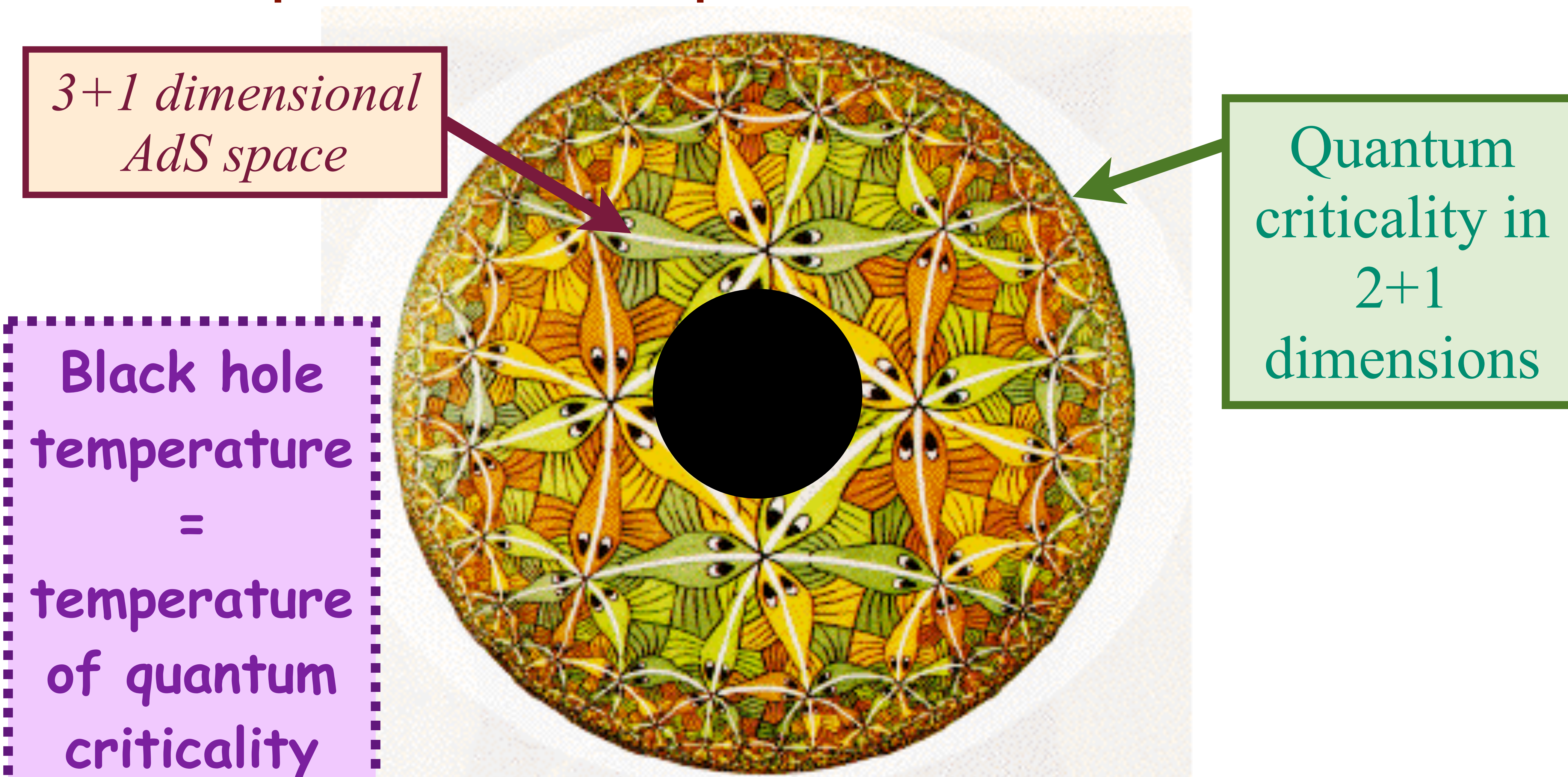

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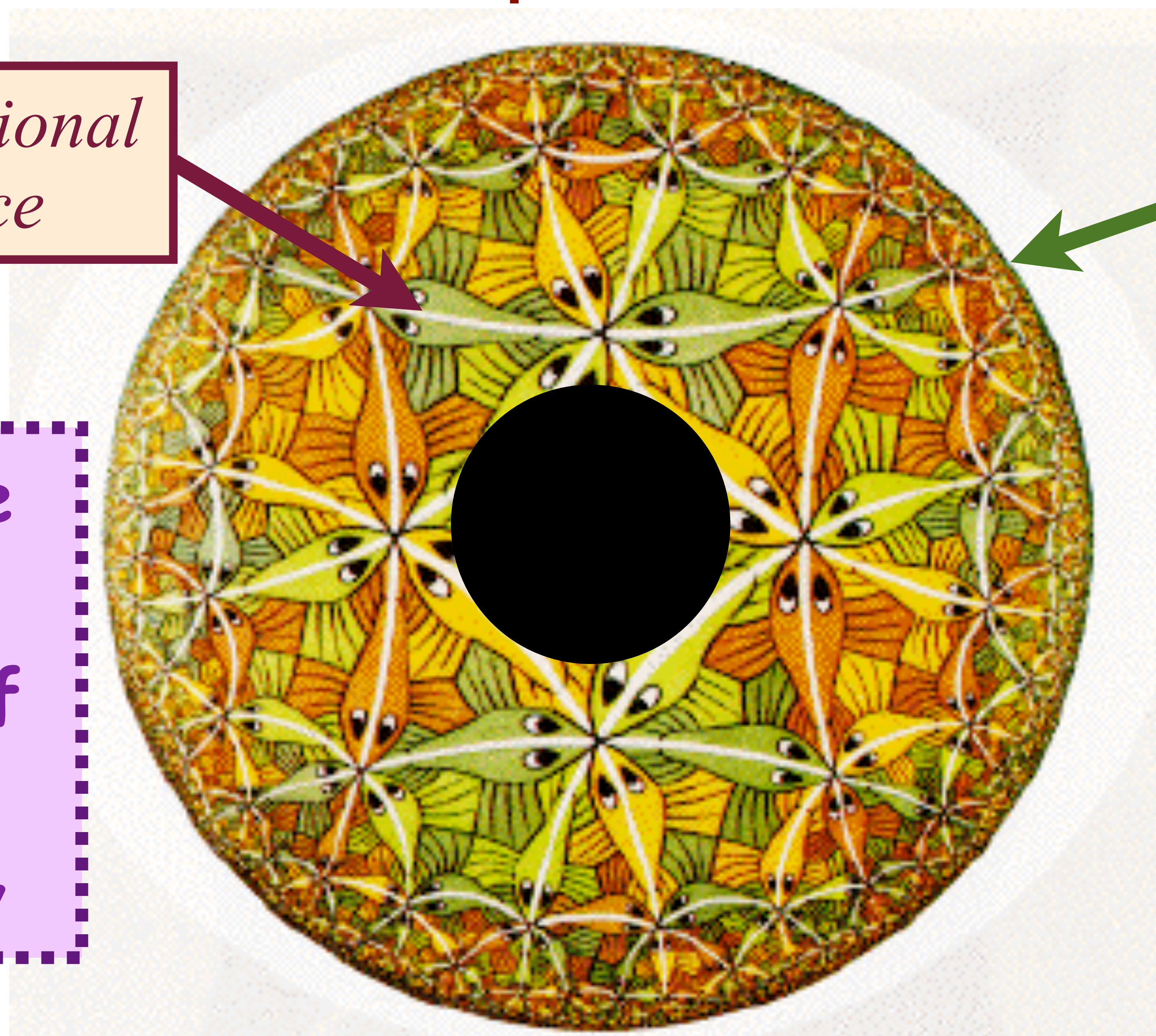

Quantum criticality in 2+1 dimensions

Black hole entropy = entropy of quantum criticality

Strominger, Vafa

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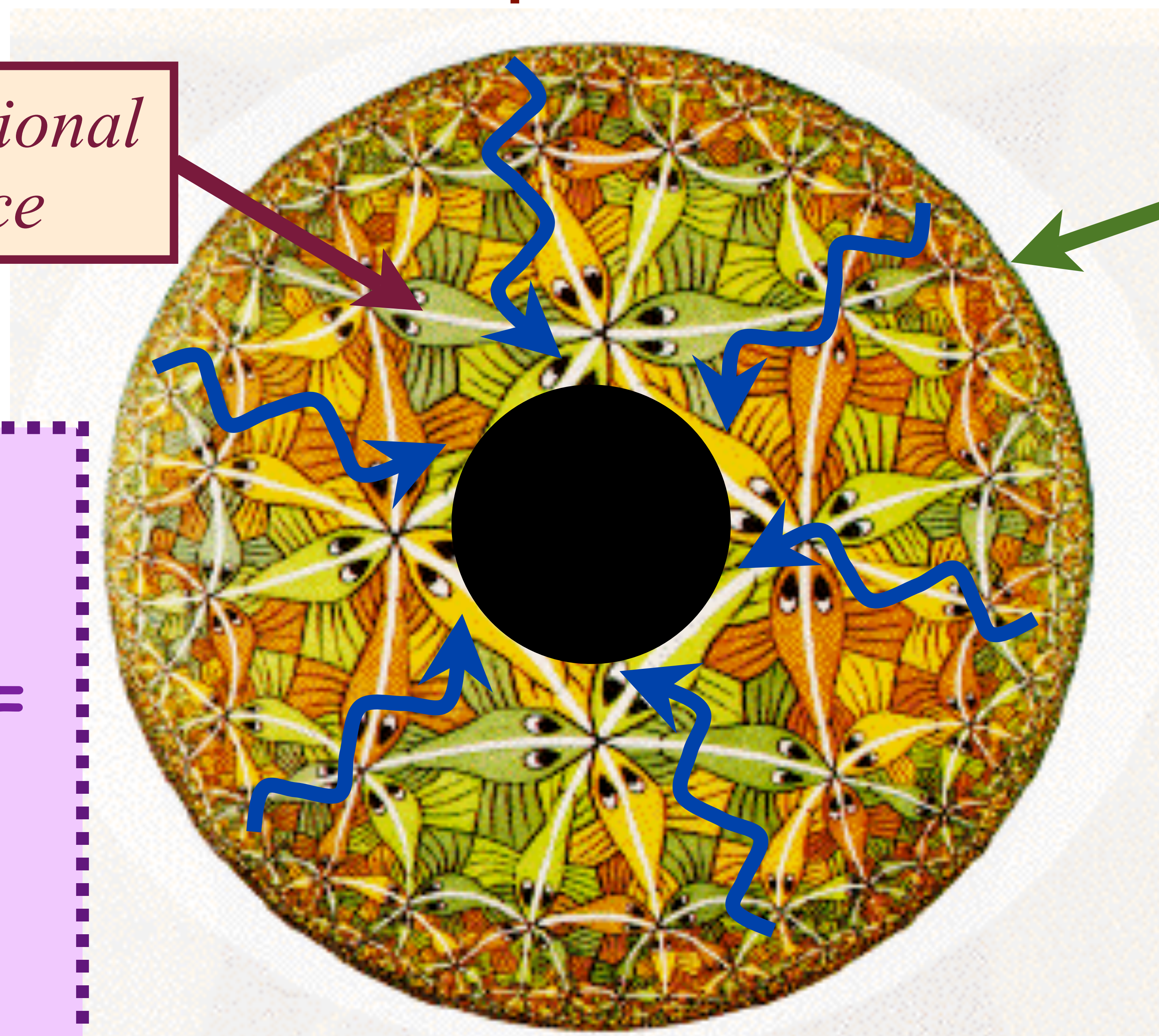
3+1 dimensional AdS space

Quantum criticality in 2+1 dimensions



Quantum critical dynamics = waves in curved space

Maldacena, Gubser, Klebanov, Polyakov, Witten

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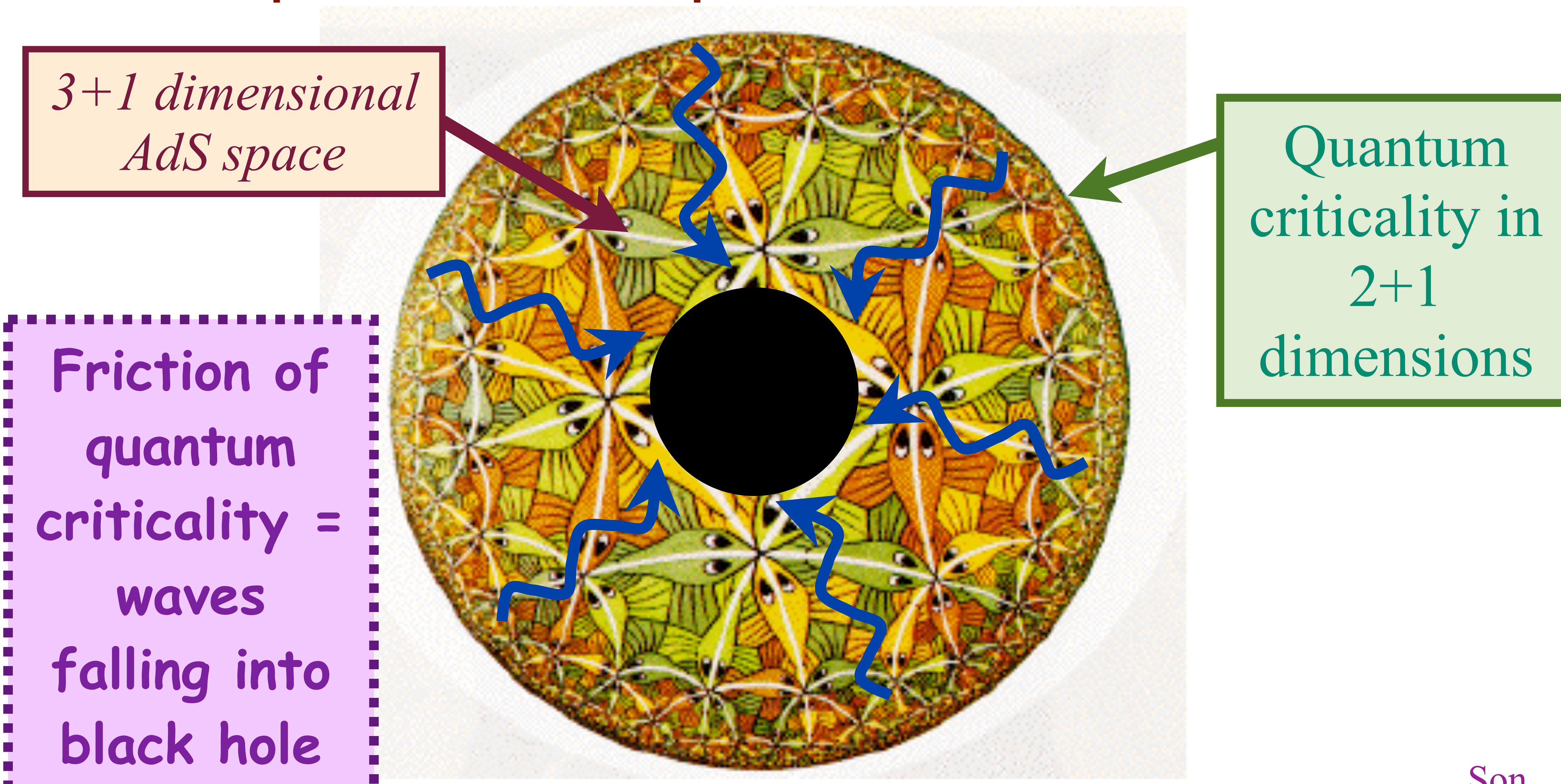


Son

SU(N) SYM3 with $\mathcal{N} = 8$ supersymmetry

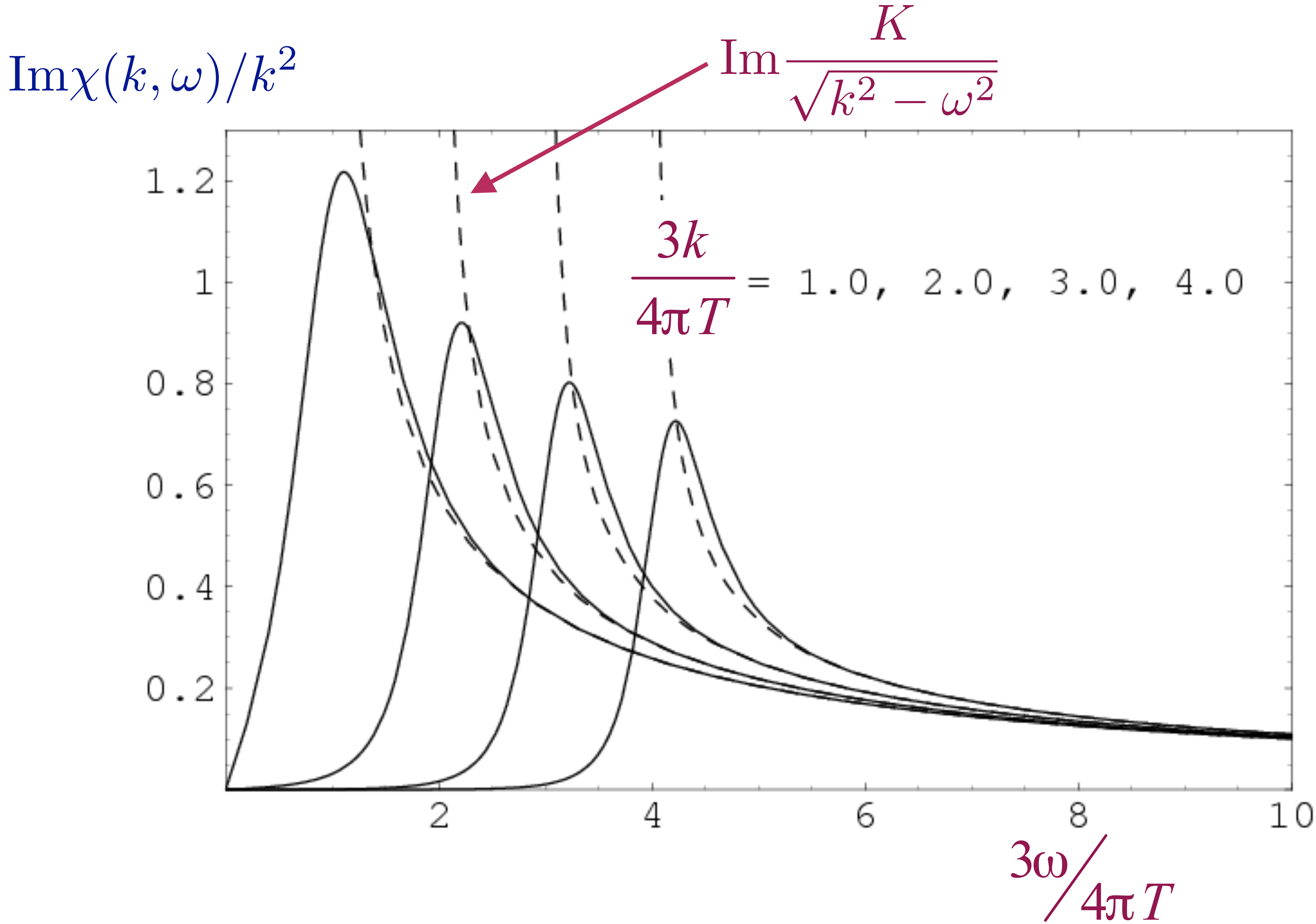
- Has a single dimensionful coupling constant, e_0, which flows to a strong-coupling fixed point $e_0 = e_0^*$ in the infrared.

- The CFT3 describing this fixed point resembles "critical spin liquid" theories.

- This CFT3 is the low energy limit of string theory on an M2 brane. The AdS/CFT correspondence provides a dual description using 11-dimensional supergravity on $\text{AdS}_4 \times S_7$.

- The CFT3 has a global SO(8) R symmetry, and correlators of the SO(8) charge density can be computed exactly in the large N limit, even at $T > 0$.

Collisionless to hydrodynamic crossover of SYM3



P. Kovtun, C. Herzog, S. Sachdev, and D.T. Son, Phys. Rev. D **75**, 085020 (2007)

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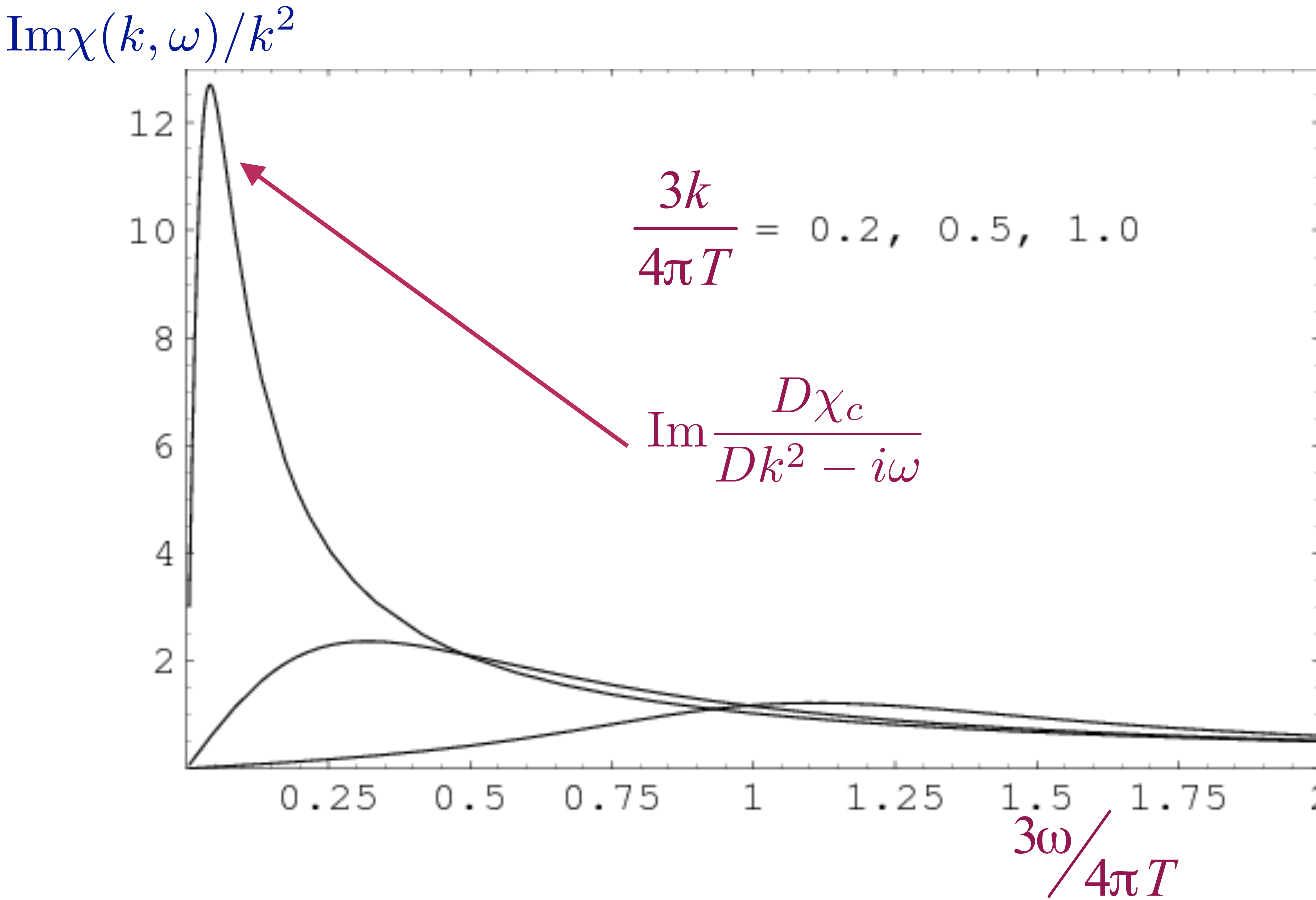

P. Kovtun, C. Herzog, S. Sachdev, and D.T. Son, Phys. Rev. D **75**, 085020 (2007)

Universal constants of SYM3

$$\chi_c = \frac{k_B T}{(hv)^2}\Theta_1$$

$$D = \frac{hv^2}{k_B T}\Theta_2$$

$$\sigma(\omega) = \begin{cases} \dfrac{4e^2}{h}K & , \quad \hbar\omega \gg k_B T \\[2ex] \dfrac{4e^2}{h}\Theta_1\Theta_2 & , \quad \hbar\omega \ll k_B T \end{cases}$$

$$\boxed{\begin{aligned} K &= \frac{\sqrt{2}N^{3/2}}{3} \\ \Theta_1 &= \frac{8\pi^2\sqrt{2}N^{3/2}}{9} \\ \Theta_2 &= \frac{3}{8\pi^2} \end{aligned}}$$

C. Herzog, JHEP **0212**, 026 (2002)
P. Kovtun, C. Herzog, S. Sachdev, and D.T. Son, Phys. Rev. D **75**, 085020 (2007)

Electromagnetic self-duality

- Unexpected result, $K = \Theta_1 \Theta_2$.

- This is traced to a *four*-dimensional electromagnetic self-duality of the theory on AdS_4. In the large N limit, the SO(8) currents decouple into 28 U(1) currents with a Maxwell action for the U(1) gauge fields on AdS_4.

- This special property is not expected for generic CFT3s.

- Open question: Does $K = \Theta_1 \Theta_2$ hold beyond the $N \to \infty$ limit ? In other words, does this "self-duality" survive in the full M theory.

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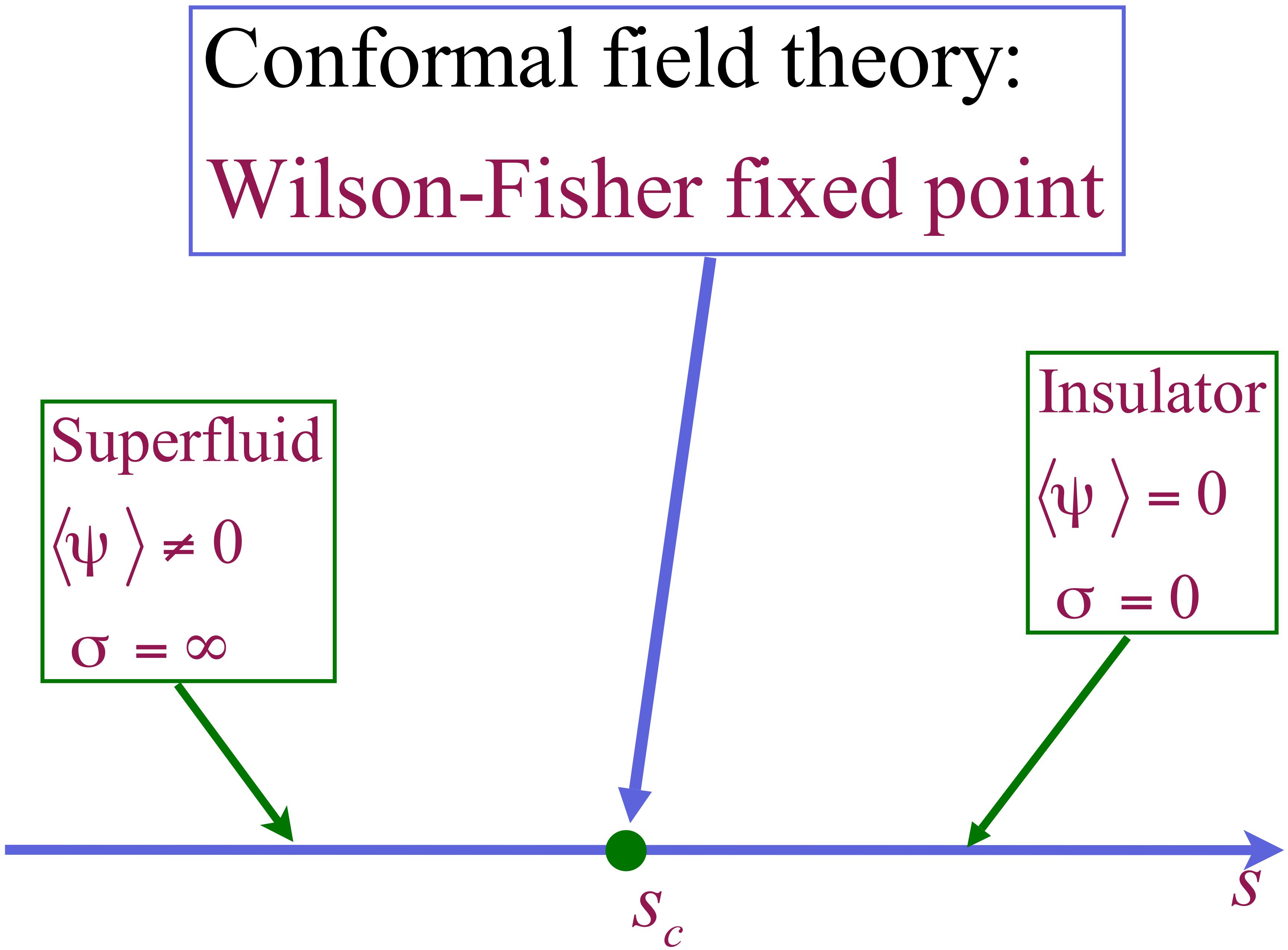
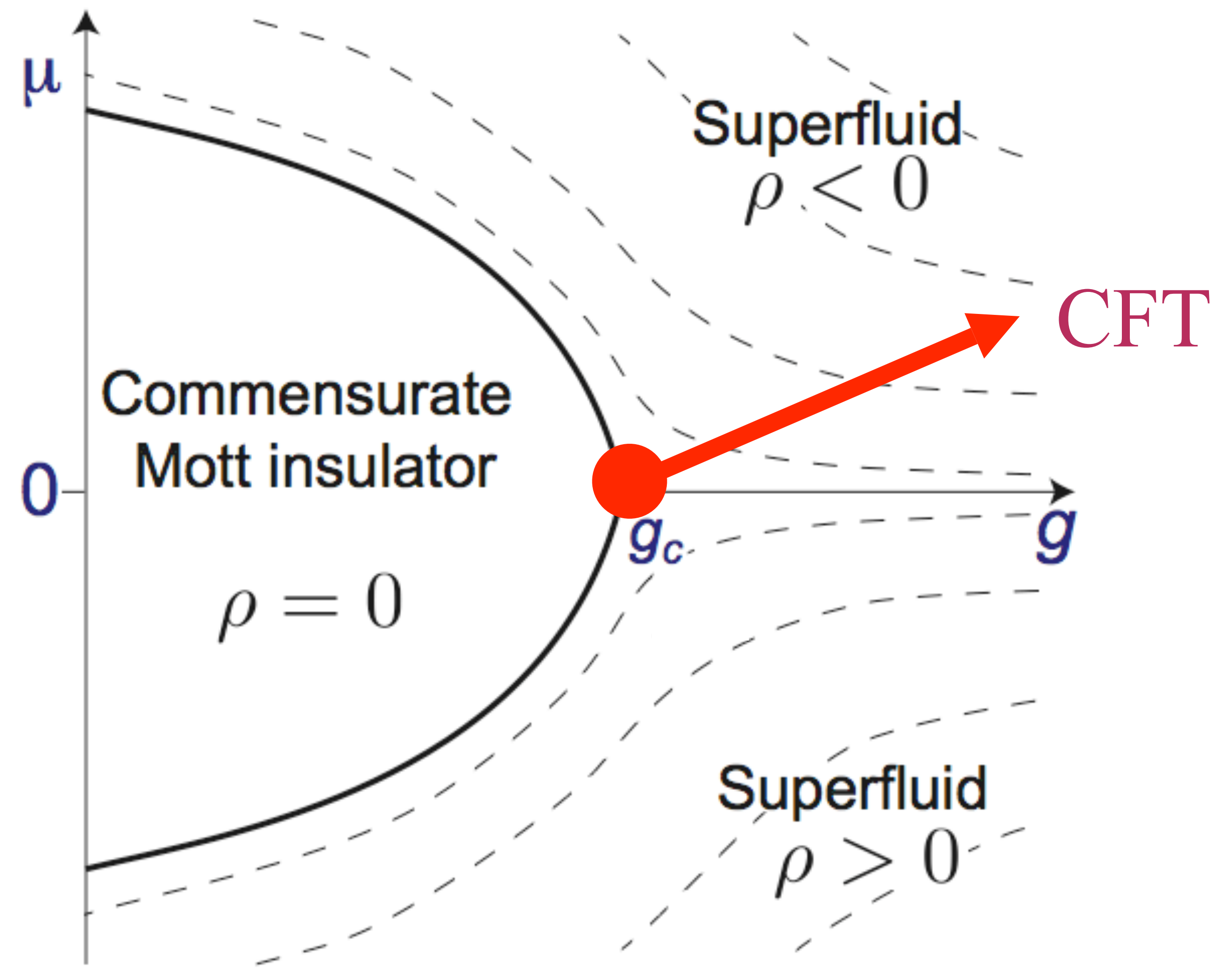


$$\mathcal{S} = \int d^2 r d\tau \left[|\partial_\tau \psi|^2 + c^2 |\vec{\nabla} \psi|^2 + s|\psi|^2 + \frac{u}{2}|\psi|^4 \right]$$

For experimental applications, we must move away from the ideal CFT

- A chemical potential μ

- A magnetic field B



e.g.

$$\mathcal{S} = \int d^2 r d\tau \left[|(\partial_\tau - \mu)\psi|^2 + v^2 \left|(\vec{\nabla} - i\vec{A})\psi\right|^2 - g|\psi|^2 + \frac{u}{2}|\psi|^4 \right]$$

$$\nabla \times \vec{A} = B$$

In the regime $\hbar\omega \ll k_B T$, we can use the principles of hydrodynamics:

- Describe system in terms of local state variables which obey the equation of state
- Express conserved currents in terms of gradients of state variables using transport co-efficients. These are restricted by demanding that the system relaxes to *local equilibrium i.e.* entropy production is positive.
- The conservation laws are the equations of motion.

The variables entering the hydrodynamic theory are

- the external magnetic field $F^{\mu\nu}$,

$$F^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & B \\ 0 & -B & 0 \end{pmatrix},$$

- $T^{\mu\nu}$, the stress energy tensor,
- J^{μ}, the current,
- ρ, the **difference** in density from the Mott insulator.
- ε, the local energy
- P, the local pressure,
- u^{μ}, the local velocity, and
- σ_Q, a universal conductivity, which is the **single transport co-efficient**.

The dependence of ε, P, σ_Q on T and v follows from simple scaling arguments

Lorentz invariance and positivity of entropy production lead to the hydrodynamic equations of motion and constitutive relations:

S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

Lorentz invariance and positivity of entropy production lead to the hydrodynamic equations of motion and constitutive relations:

$$\begin{aligned} \partial_\mu J^\mu &= 0 \\ \partial_\mu T^{\mu\nu} &= F^{\mu\nu} J_\nu \end{aligned}$$

Conservation laws/equations of motion

S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

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\begin{aligned}
\partial_\mu J^\mu &= 0 \\
\partial_\mu T^{\mu\nu} &= F^{\mu\nu} J_\nu \\
T^{\mu\nu} &= (\varepsilon + P) u^\mu u^\nu + P g^{\mu\nu} \\
J^\mu &= \rho u^\mu
\end{aligned}
$$

Constitutive relations which follow from Lorentz transformation to moving frame

S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

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T^{\mu\nu} &= (\varepsilon + P) u^\mu u^\nu + P g^{\mu\nu} \\
J^\mu &= \rho u^\mu + \sigma_Q (g^{\mu\nu} + u^\mu u^\nu) \left[(-\partial_\nu \mu + F_{\nu\lambda} u^\lambda) + \mu \frac{\partial_\mu T}{T} \right]
\end{aligned}
$$

Single dissipative term allowed by requirement of positive entropy production. There is only one independent transport co-efficient

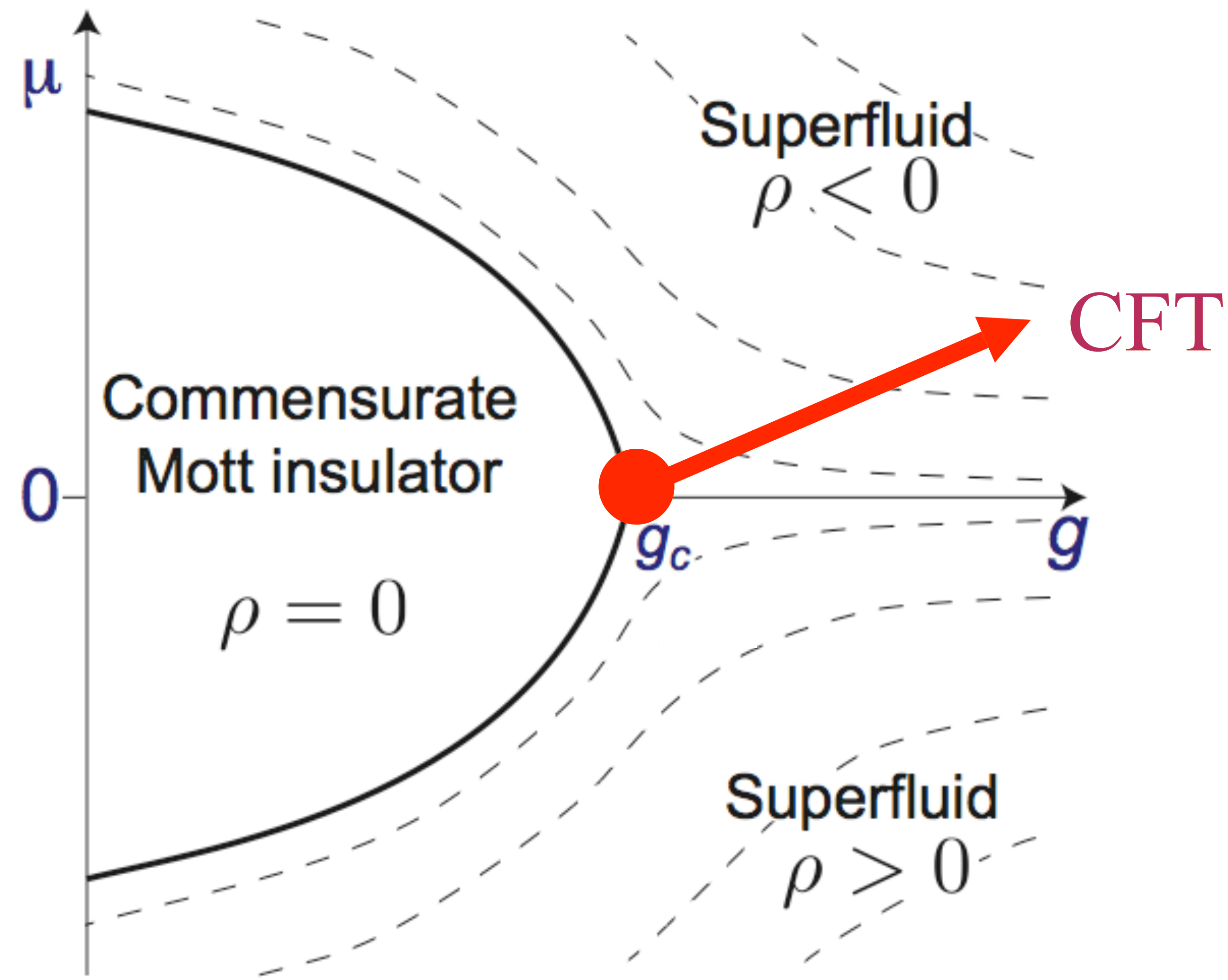
S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

For experimental applications, we must move away from the ideal CFT

- A chemical potential μ

- A magnetic field B

- An impurity scattering rate $1/\tau_{imp}$ (its T dependence follows from scaling arguments)



e.g.

$$\mathcal{S} = \int d^2 r d\tau \left[|(\partial_\tau - \mu)\psi|^2 + v^2 \left| (\vec{\nabla} - i\vec{A})\psi \right|^2 - g|\psi|^2 + V(r)|\psi|^2 + \frac{u}{2}|\psi|^4 \right]$$

$$\nabla \times \vec{A} = B \quad , \quad \overline{V(r)} = 0 \quad , \quad \overline{V(r)V(r')} = V_{\text{imp}}^2 \delta^2(r - r')$$

Lorentz invariance and positivity of entropy production lead to the hydrodynamic equations of motion and constitutive relations:

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\partial_\mu J^\mu &= 0 \\
\partial_\mu T^{\mu\nu} &= F^{\mu\nu} J_\nu + \frac{1}{\tau_{\text{imp}}} \left(\delta^\mu_\nu + u^\mu u_\nu\right) T^{\nu\gamma} u_\gamma \\
T^{\mu\nu} &= (\varepsilon + P) u^\mu u^\nu + P g^{\mu\nu} \\
J^\mu &= \rho u^\mu + \sigma_Q (g^{\mu\nu} + u^\mu u^\nu) \left[\left(-\partial_\nu \mu + F_{\nu\lambda} u^\lambda\right) + \mu \frac{\partial_\mu T}{T} \right]
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$$

Solve initial value problem and relate
results to response functions (Kadanoff+Martin)

S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

From these relations, we obtained results for the transport co-efficients, expressed in terms of a "cyclotron" frequency and damping:

$$\omega_c = \frac{2eB\rho v^2}{c(\varepsilon + P)} \quad , \quad \gamma = \sigma_Q \frac{B^2 v^2}{c^2(\varepsilon + P)}$$

Longitudinal conductivity

$$\sigma_{xx} = \sigma_Q \left[\frac{(\omega + i/\tau_{\text{imp}})(\omega + i\gamma + i\omega_c^2/\gamma + i/\tau_{\text{imp}})}{(\omega + i\gamma + i/\tau_{\text{imp}})^2 - \omega_c^2} \right].$$

S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

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$$= \sigma_Q + \frac{4e^2\rho^2 v^2}{(\varepsilon + P)} \frac{1}{(-i\omega + 1/\tau_{\rm imp})} \qquad \text{as } B \to 0$$

S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

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Hall conductivity

$$\sigma_{xy} = -\frac{2e\rho c}{B}\left[\frac{\gamma^2 + \omega_c^2 - 2i\gamma\omega + 2\gamma/\tau_{\text{imp}}}{(\omega + i\gamma + i/\tau_{\text{imp}})^2 - \omega_c^2}\right]$$

S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

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S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

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S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

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Thermal conductivity

$$\begin{aligned}\kappa_{xx} &= \sigma_Q \left(\frac{k_B^2 T}{4e^2}\right)\left(\frac{\varepsilon + P}{k_B T \rho}\right)^2 \left[\frac{(\omega_c^2/\gamma)(\omega_c^2/\gamma + 1/\tau_{\text{imp}})}{(\omega_c^2/\gamma + 1/\tau_{\text{imp}})^2 + \omega_c^2}\right] \\ &= \frac{1}{\sigma_Q} k_B^2 T \left(\frac{c(\varepsilon + P)}{k_B T B}\right)^2 \left[\frac{\gamma(\omega_c^2/\gamma + 1/\tau_{\text{imp}})}{(\omega_c^2/\gamma + 1/\tau_{\text{imp}})^2 + \omega_c^2}\right]\end{aligned}$$

S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

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Thermal conductivity

$$\kappa_{xx} = \sigma_Q \left(\frac{k_B^2 T}{4e^2}\right)\left(\frac{\varepsilon + P}{k_B T \rho}\right)^2 \quad \boxed{\to 1 \text{ as } B \to 0}$$

$$= \frac{1}{\sigma_Q} k_B^2 T \left(\frac{c(\varepsilon + P)}{k_B T B}\right)^2 \left[\frac{\gamma(\omega_c^2/\gamma + 1/\tau_{\text{imp}})}{(\omega_c^2/\gamma + 1/\tau_{\text{imp}})^2 + \omega_c^2}\right]$$

S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

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S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

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$$\omega_c = \frac{2eB\rho v^2}{c(\varepsilon + P)} \quad , \quad \gamma = \frac{B^2 v^2}{c^2(\varepsilon + P)}$$

Nernst signal

$$e_N = \left(\frac{k_B}{2e}\right)\left(\frac{\varepsilon + P}{k_B T \rho}\right)\left[\frac{\omega_c/\tau_{\mathrm{imp}}}{(\omega_c^2/\gamma + 1/\tau_{\mathrm{imp}})^2 + \omega_c^2}\right]$$

$$\frac{k_B}{2e} = 43.086 \mu\mathrm{V}/\mathrm{K}$$

S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

Exact Results

To the solvable supersymmetric, Yang-Mills theory CFT, we add

- A chemical potential μ
- A magnetic field B

After the AdS/CFT mapping, we obtain the Einstein-Maxwell theory of a black hole with

- An electric charge
- A magnetic charge

The exact results are found to be in *precise* accord with *all* hydrodynamic results presented earlier

S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

Solve quantum Boltzmann equation for graphene

The results are found to be in *precise* accord with *all* hydrodynamic results presented earlier, and many results are extended beyond hydrodynamic regime.

M. Müller, L. Fritz, and S. Sachdev, *arXiv:0805.1413*

Collisionless-hydrodynamic crossover in pure, undoped, graphene

$$\sigma_Q(\omega) = \begin{cases} \dfrac{e^2}{h}\left[\dfrac{\pi}{2} + \mathcal{O}\left(\dfrac{1}{\ln(\Lambda/\omega)}\right)\right] & , \quad \hbar\omega \gg k_B T \\ \dfrac{e^2}{h\alpha^2(T)}\left[0.760 + \mathcal{O}\left(\dfrac{1}{|\ln(\alpha(T))|}\right)\right] & , \quad \hbar\omega \ll k_B T \alpha^2(T) \end{cases}$$

I. Herbut, V. Juricic, and O. Vafek, *Phys. Rev. Lett.* **100**, 046403 (2008).

where $\alpha(T)$ is the T-dependent fine structure constant which obeys

$$\alpha(T) = \frac{\alpha}{1 + (\alpha/4)\ln(\Lambda/T)} \overset{T\to 0}{\sim} \frac{4}{\ln(\Lambda/T)}$$

L. Fritz, M. Mueller, J. Schmalian and S. Sachdev, arXiv:0802.4289

See also A. Kashuba, arXiv:0802.2216

1. CFT3s in condensed matter physics and string theory

   Superfluid-insulator transition, magnetic ordering transitions, graphene ....

2. Quantum-critical transport

   Collisionless-to-hydrodynamic crossover of CFT3s

3. Black Hole Thermodynamics

   Connections to quantum criticality

4. Generalized magnetohydrodynamics

   Quantum criticality and dyonic black holes

5. Experiments

   Graphene and the cuprate superconductors

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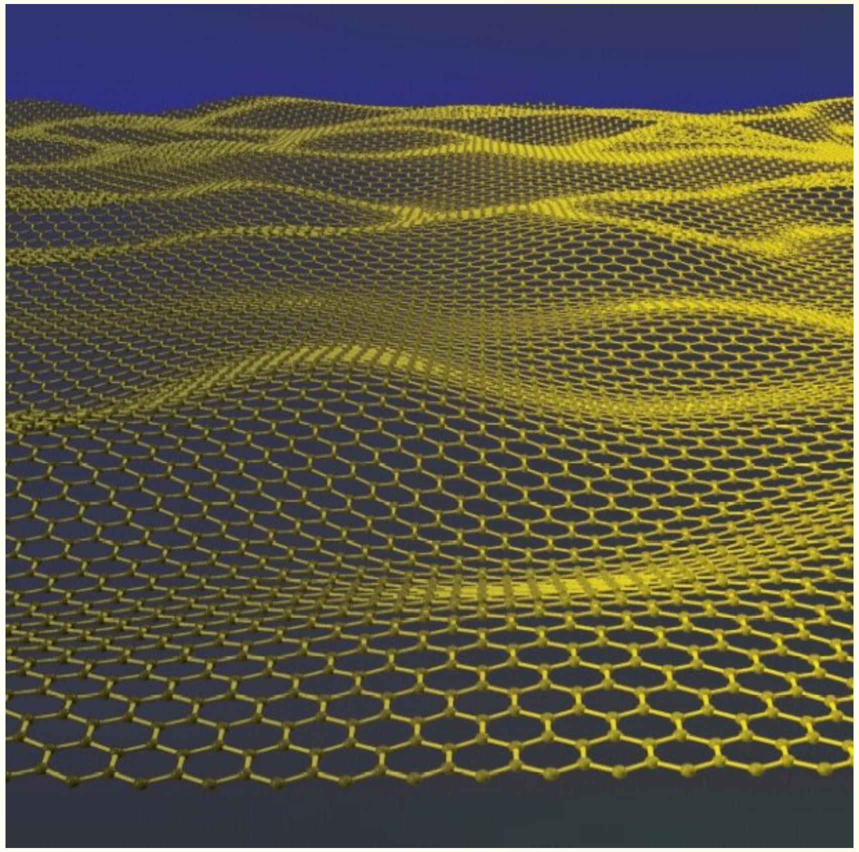
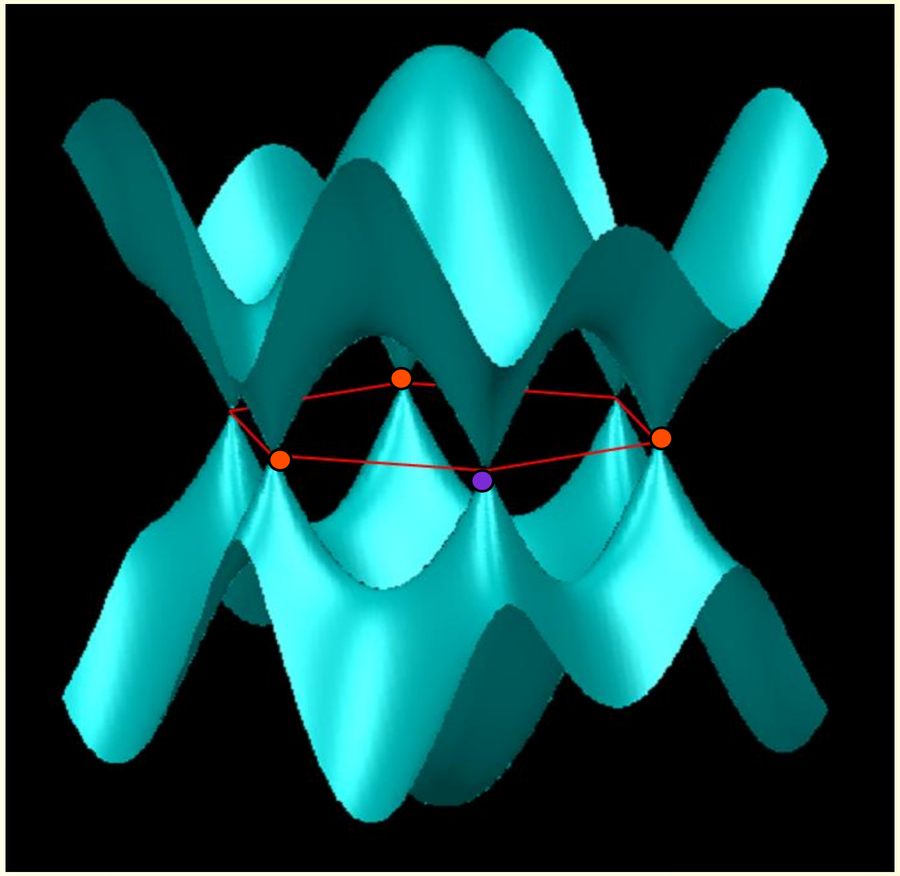
Dirac fermions in graphene

Honeycomb lattice of C atoms



Tight binding dispersion



Close to the two Fermi points **K**, **K'**:

2 relativistic (Dirac) cones in the Brillouin zone

$$H \approx \mathrm{v}_F \, (\mathbf{p} - \mathbf{K}) \cdot \sigma_{\text{sublattice}}$$

$$® \quad E_{\mathbf{k}} = \mathrm{v}_F |\mathbf{k} - \mathbf{K}|$$

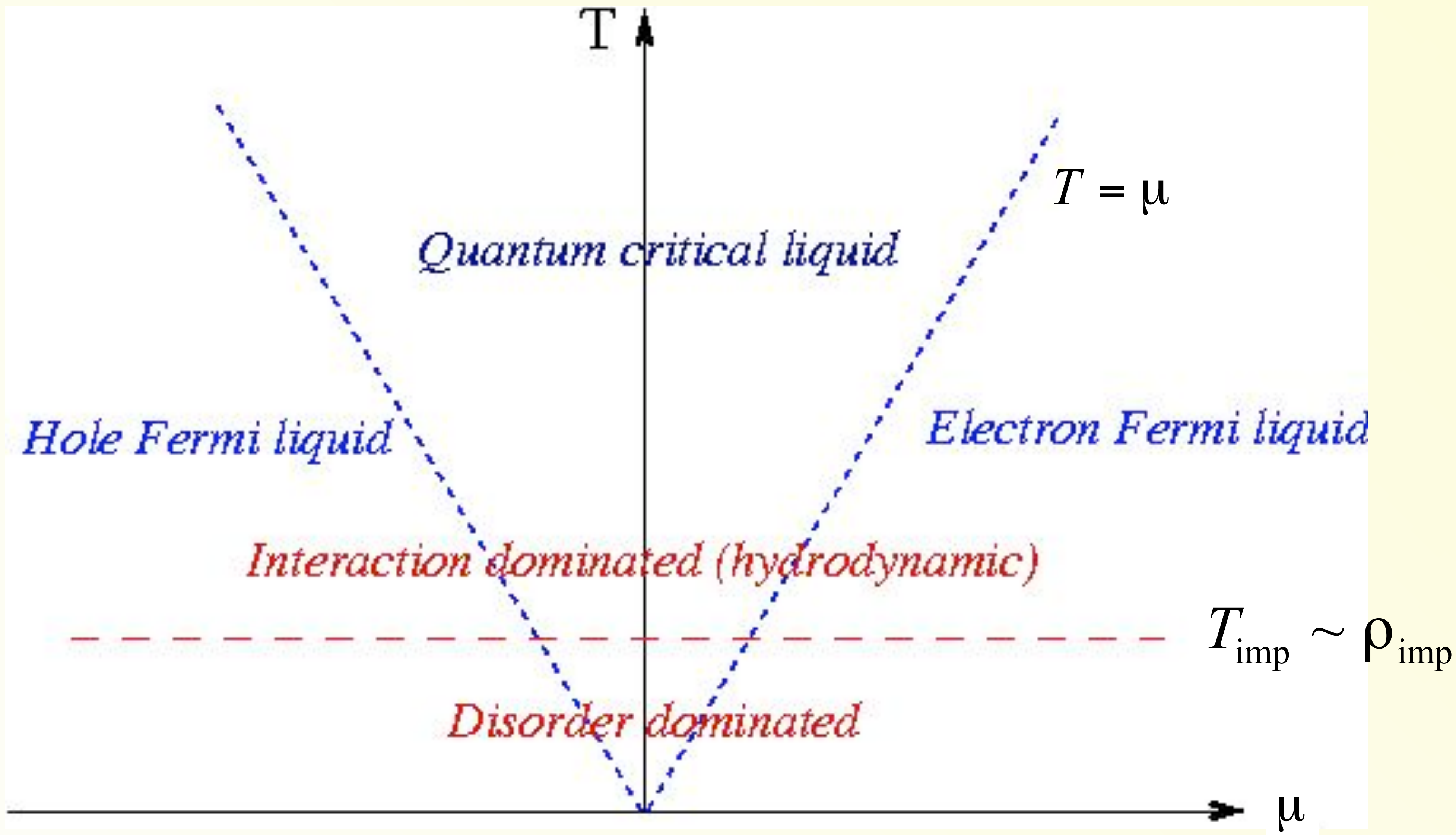
"Speed of light"

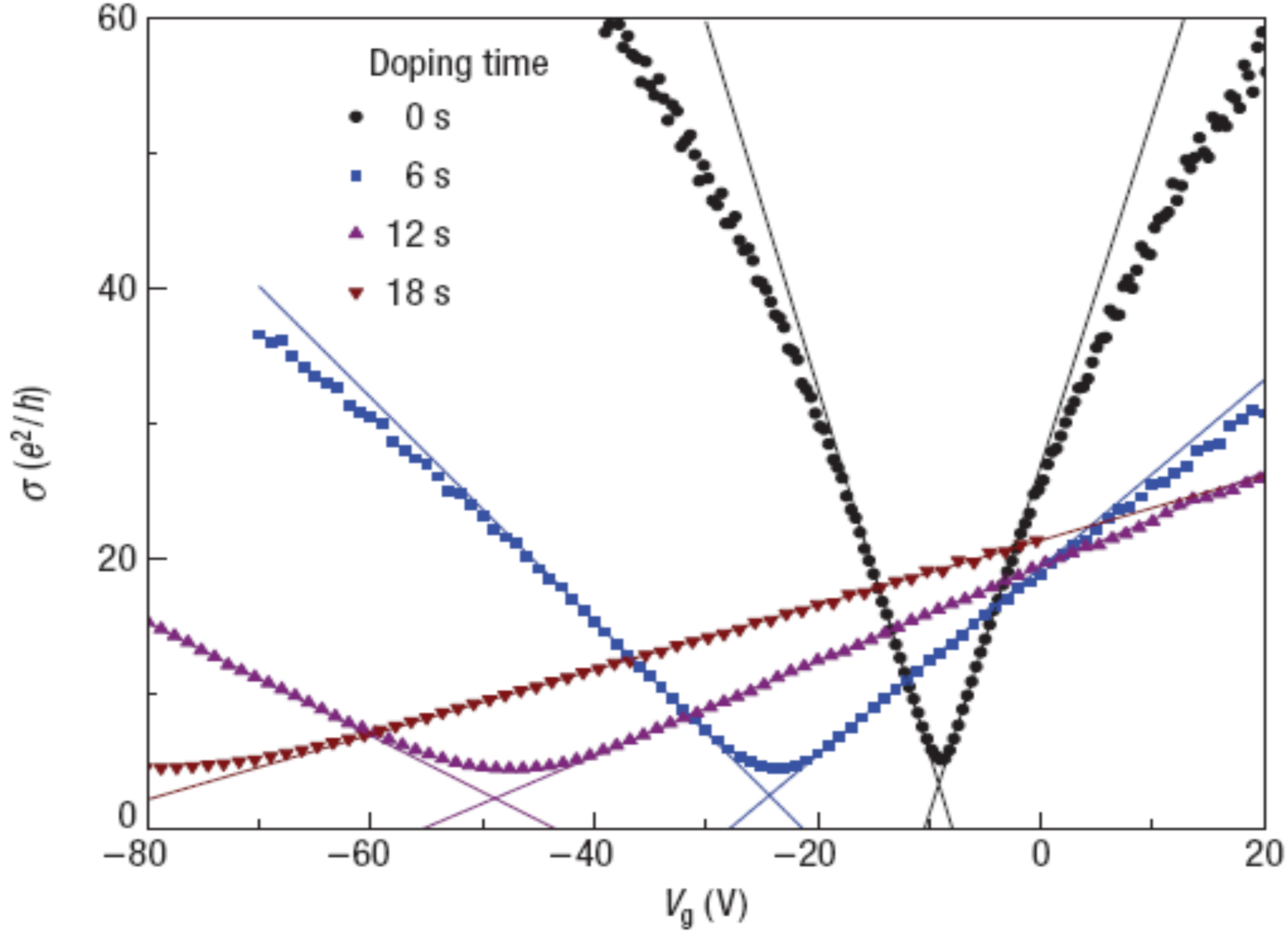
$$\mathrm{v}_F \approx 1.1 \times 10^6 \text{ m/s} \approx \frac{c}{300}$$

Coulomb interactions: Fine structure constant

$$\alpha \equiv \frac{e^2}{\varepsilon_r \hbar \mathrm{v}_F} = O(1)$$

Phase diagram & Quantum criticality





J.-H. Chen et al. Nat. Phys. ***4****, 377 (2008).*

Universal conductivity σ_Q : graphene

L. Fritz, J. Schmalian, M. Mueller, and S. Sachdev, arXiv:0802.4289

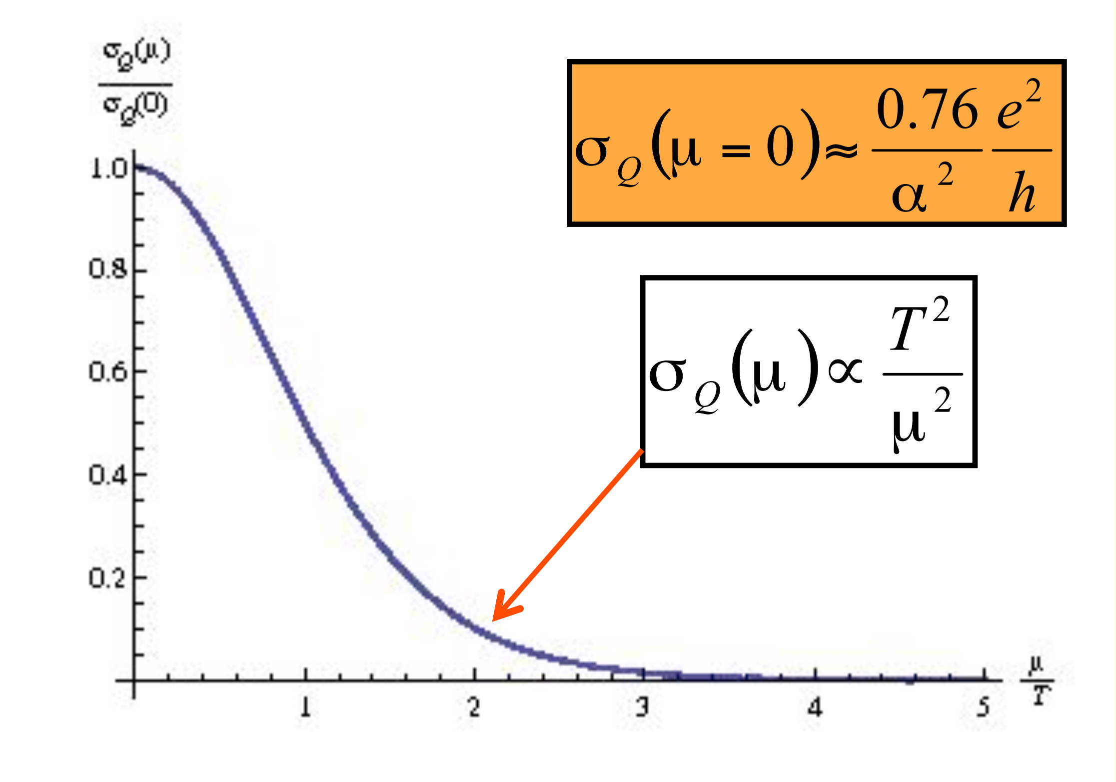
General doping:

Clean system:

$$\sigma_{xx}(\omega;\mu,\Delta=0)=e^2\frac{\rho^2 v_F^2}{\varepsilon+P}\frac{1}{(-i\omega)}+\sigma_Q$$

$$\boxed{\sigma_Q(\mu,\omega)=\frac{e^2}{\hbar}\frac{1}{\alpha^2}\frac{2\hat{g}_1}{N}\left[I_+^{(1)}-\frac{\rho^2(\hbar v)^2}{(\varepsilon+P)T}\right]^2\frac{1}{1-i\omega\tau_{ee}}}$$

Gradual disappearance of quantum criticality and relativistic physics



Will appear in all Boltzmann formulae below!

Universal conductivity σ_Q : graphene

L. Fritz, J. Schmalian, M. Mueller, and S. Sachdev, arXiv:0802.4289

General doping:

Lightly disordered system:

$$\sigma_{xx}(\omega;\mu,\Delta) = \frac{e^2}{\tau_{\rm imp}^{-1} - i\omega} \frac{\rho^2 v_F^2}{\varepsilon + P} + \sigma_Q + \delta\sigma(\Delta,\omega,\mu)$$

$$\delta\sigma(\Delta,\omega,\mu) = \mathcal{O}(\Delta/\alpha^2)$$

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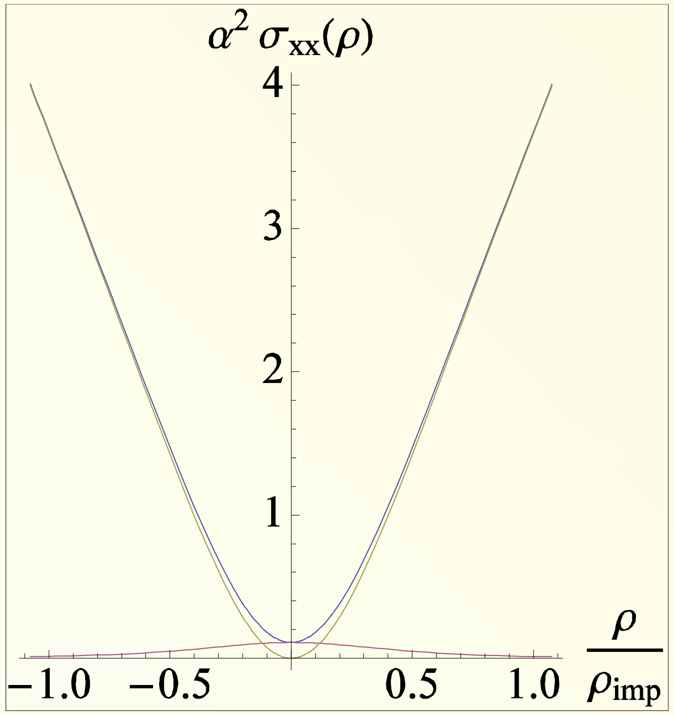
Fermi liquid regime:

$$\sigma_{xx}(\omega = 0;\mu \gg T) \approx \frac{e^2\rho^2 v_F^2 \tau_{\rm imp}}{\varepsilon + P} = \frac{2}{\pi}\frac{1}{(Z\alpha)^2}\frac{e^2}{h}\frac{\rho}{\rho_{\rm imp}}$$



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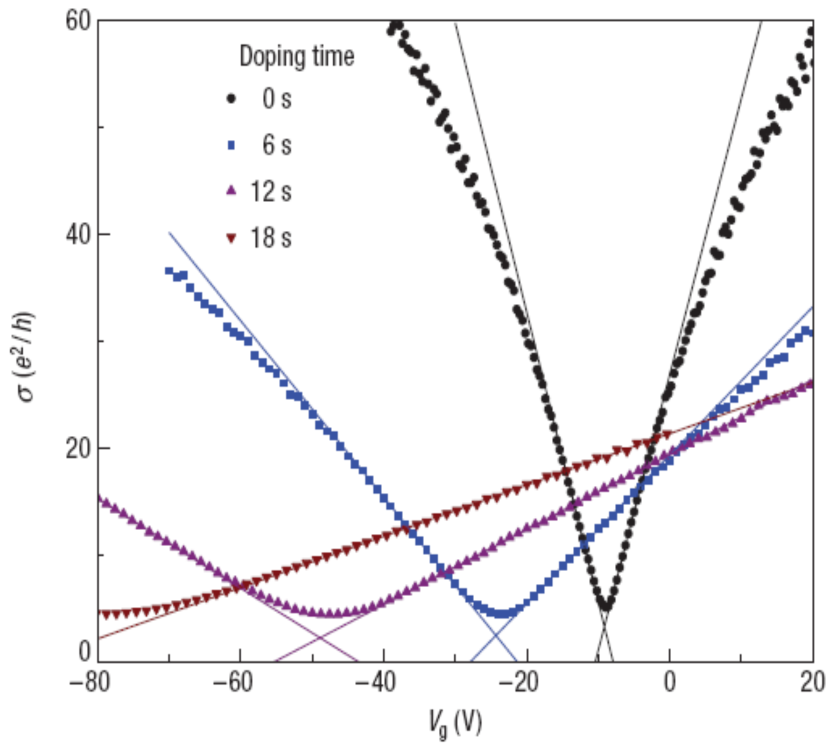
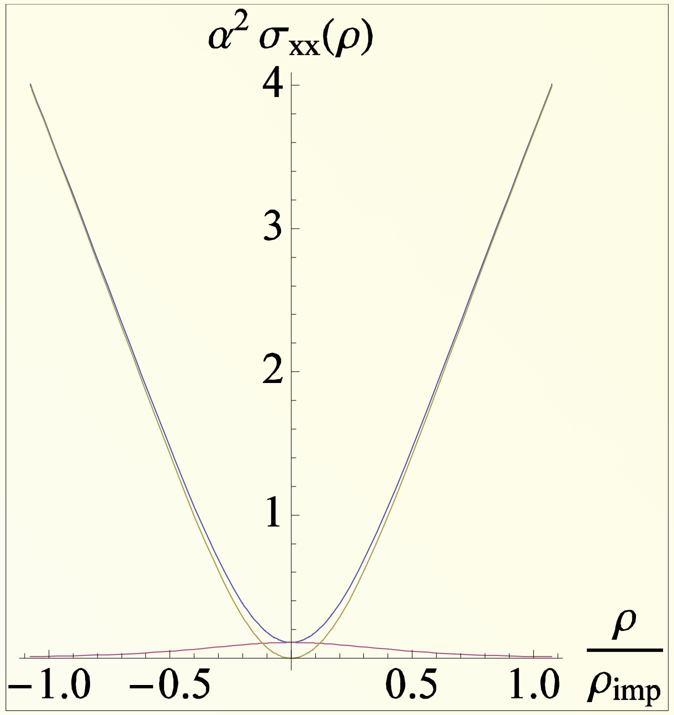
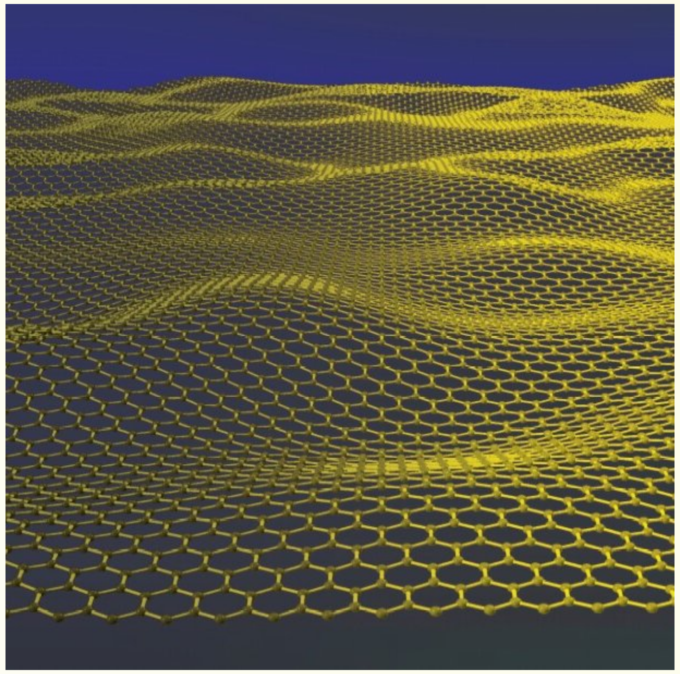
J.-H. Chen et al. Nat. Phys. ***4****, 377 (2008).*





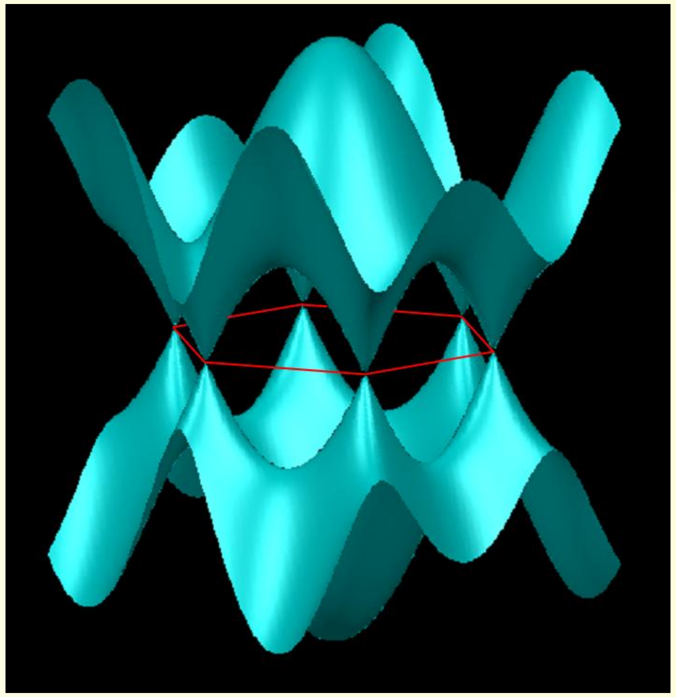
Cyclotron resonance in graphene

M. Mueller, and S. Sachdev, arXiv:0801.2970.



$$\omega = \pm\omega_c^{rel} - i\gamma - i/\tau$$

$$v = 1.1\times 10^6\, m/s \approx c/300$$



Conditions to observe resonance

- Negligible Landau quantization $\quad E_{LL} = \hbar v\sqrt{\frac{2eB}{\hbar c}} << k_B T$
- Hydrodynamic, collison-dominated regime $\quad \hbar\omega_c^{rel} << k_B T$
- Negligible broadening $\quad \gamma, \tau^{-1} < \omega_c^{rel}$
- Relativistic, quantum critical regime $\quad \rho \leq \rho_{th} = \frac{(k_B T)^2}{(\hbar v)^2}$

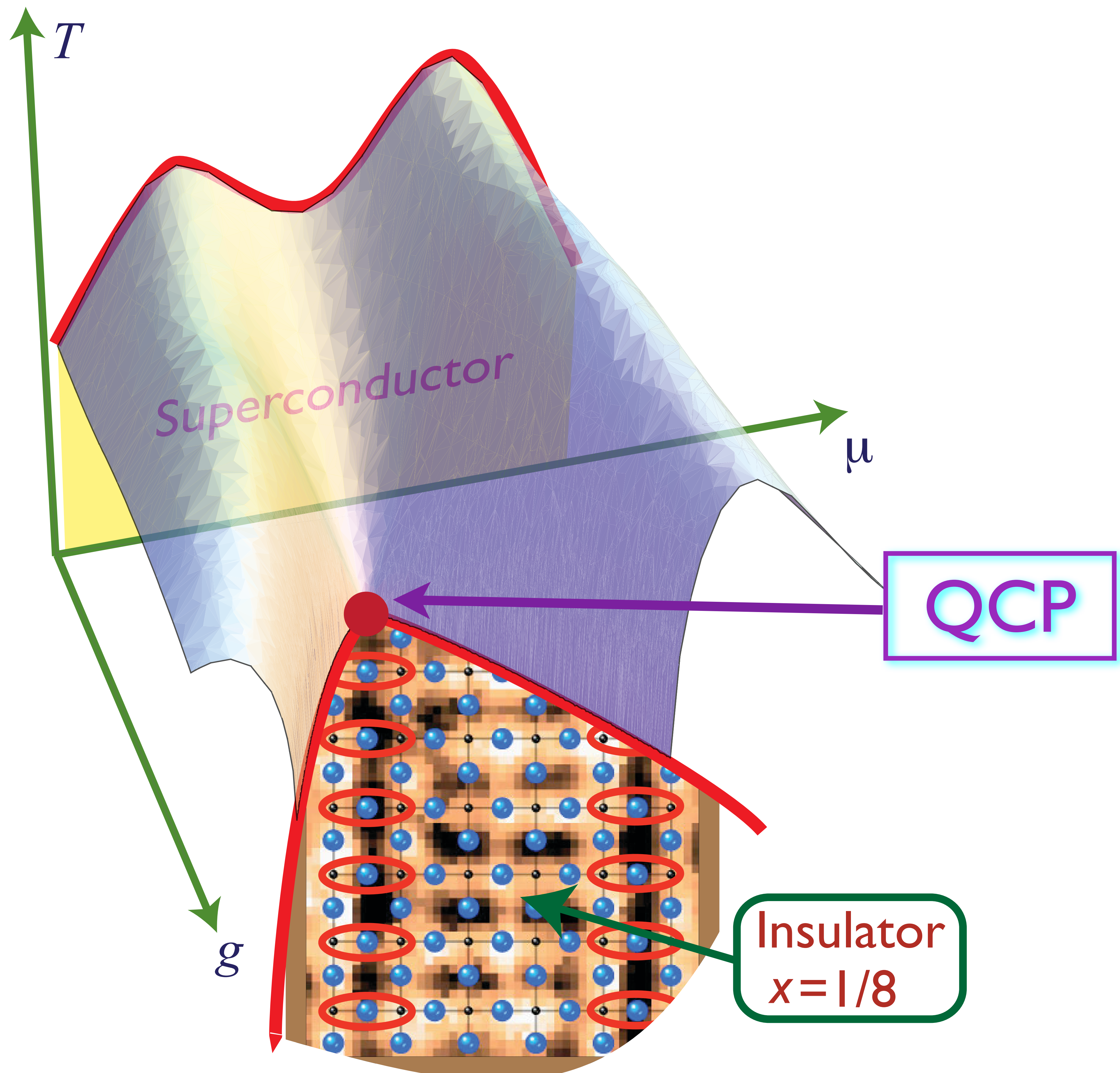
$T \approx 300\,\mathrm{K}$

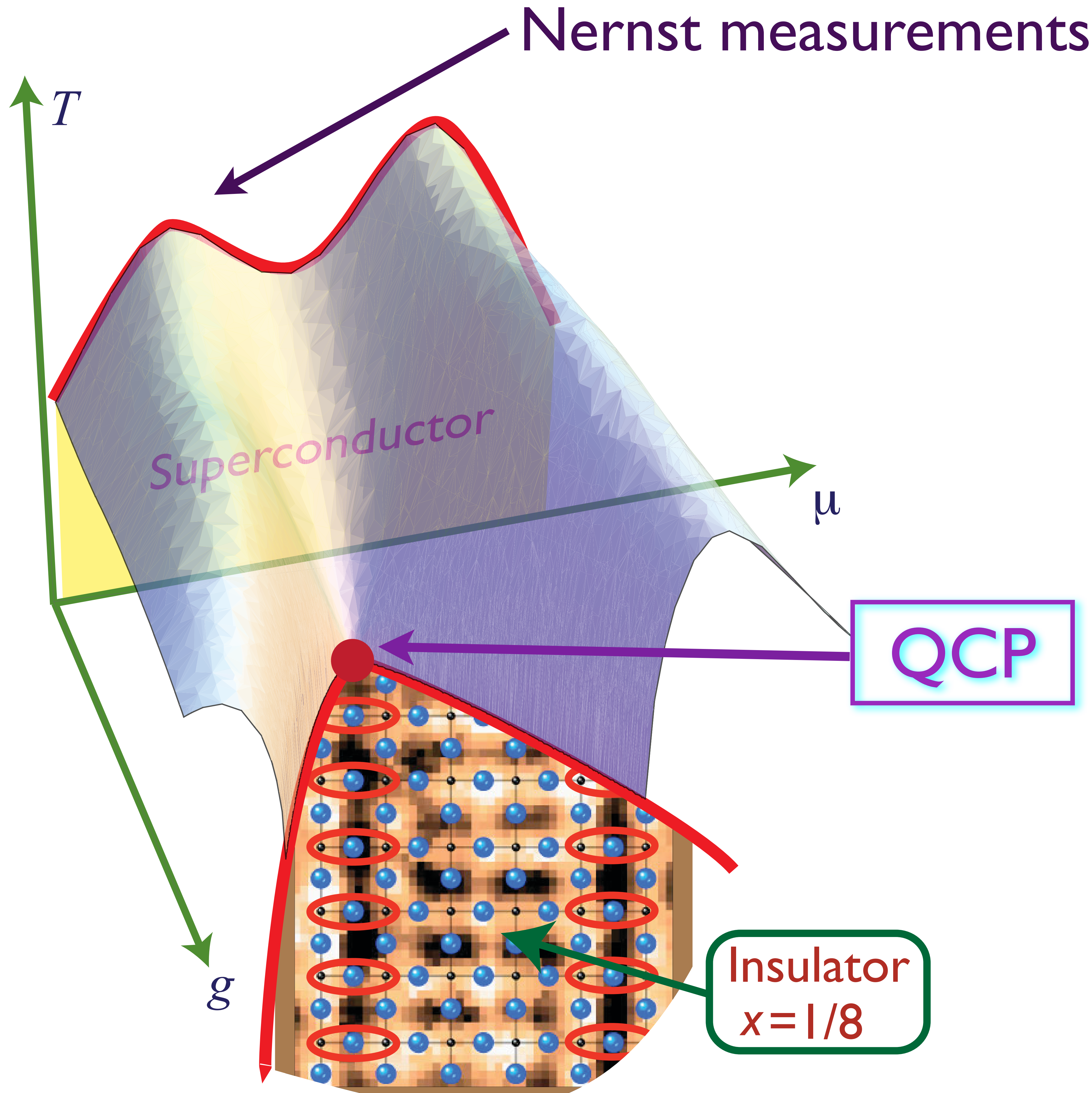
$B \approx 0.1\,\mathrm{T}$

$\rho \approx 10^{11}\,\mathrm{cm}^{-2}$

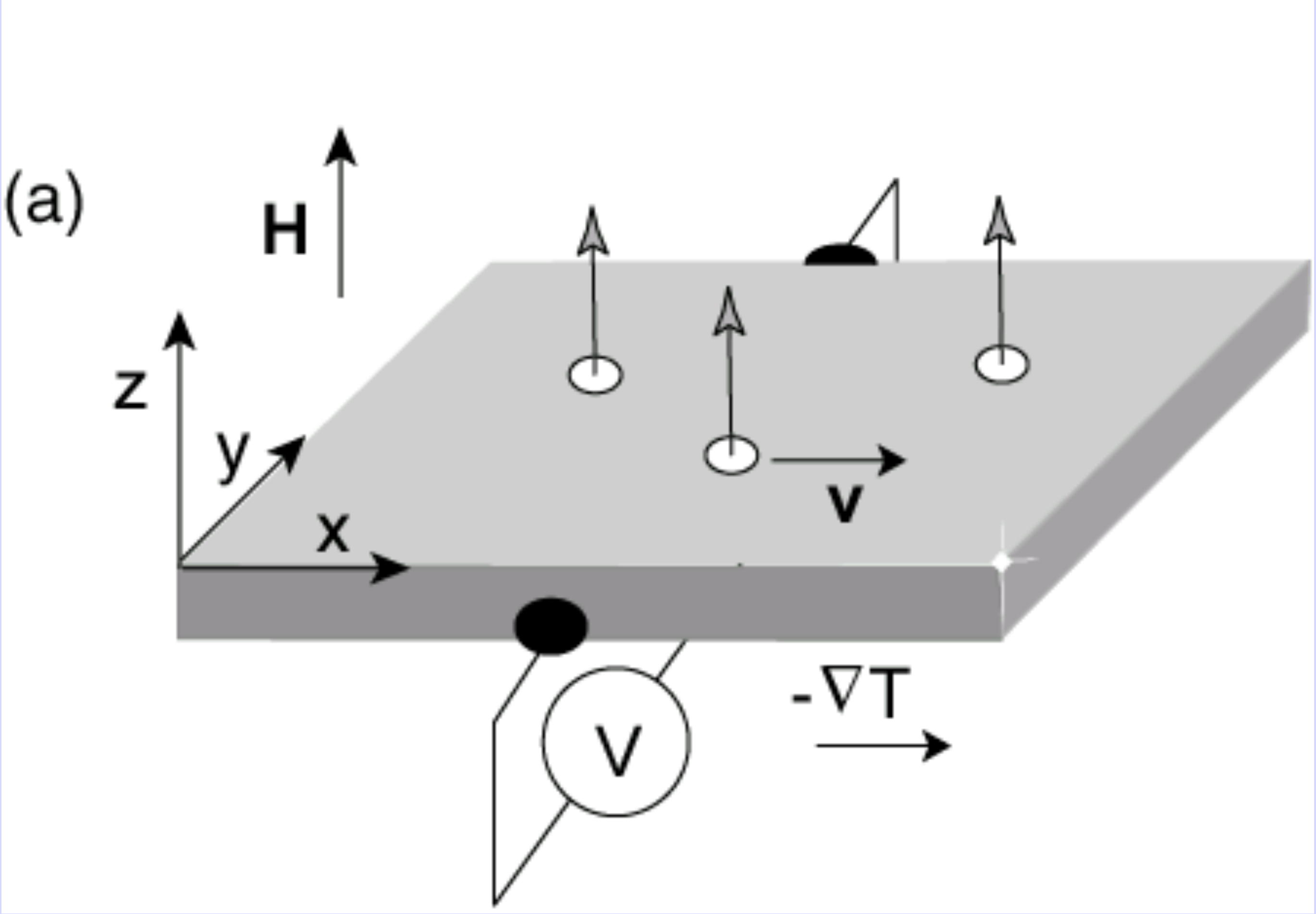
$\omega_c \approx 10^{13}\,\mathrm{s}^{-1}$

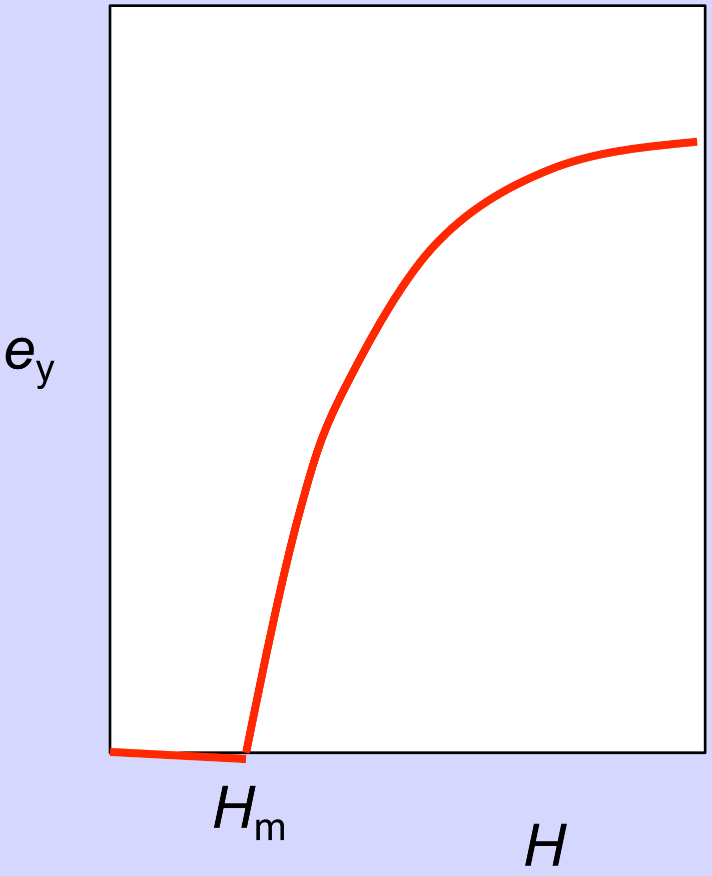
Proposed phase diagram for cuprates




Nernst measurements
T
Superconductor
μ
QCP
g
Insulator
x=1/8

Nernst experiment





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Transverse thermoelectric co-efficient

$$\left(\frac{h}{2ek_B}\right)\alpha_{xy} = \Phi_s \overline{B}\,(k_B T)^2 \left(\frac{2\pi\tau_{\rm imp}}{\hbar}\right)^2 \frac{\overline{\rho}^2 + \Phi_\sigma \Phi_{\varepsilon+P} (k_B T)^3\,\hbar/2\pi\tau_{\rm imp}}{\Phi_{\varepsilon+P}^2 (k_B T)^6 + \overline{B}^2 \overline{\rho}^2 (2\pi\tau_{\rm imp}/\hbar)^2},$$

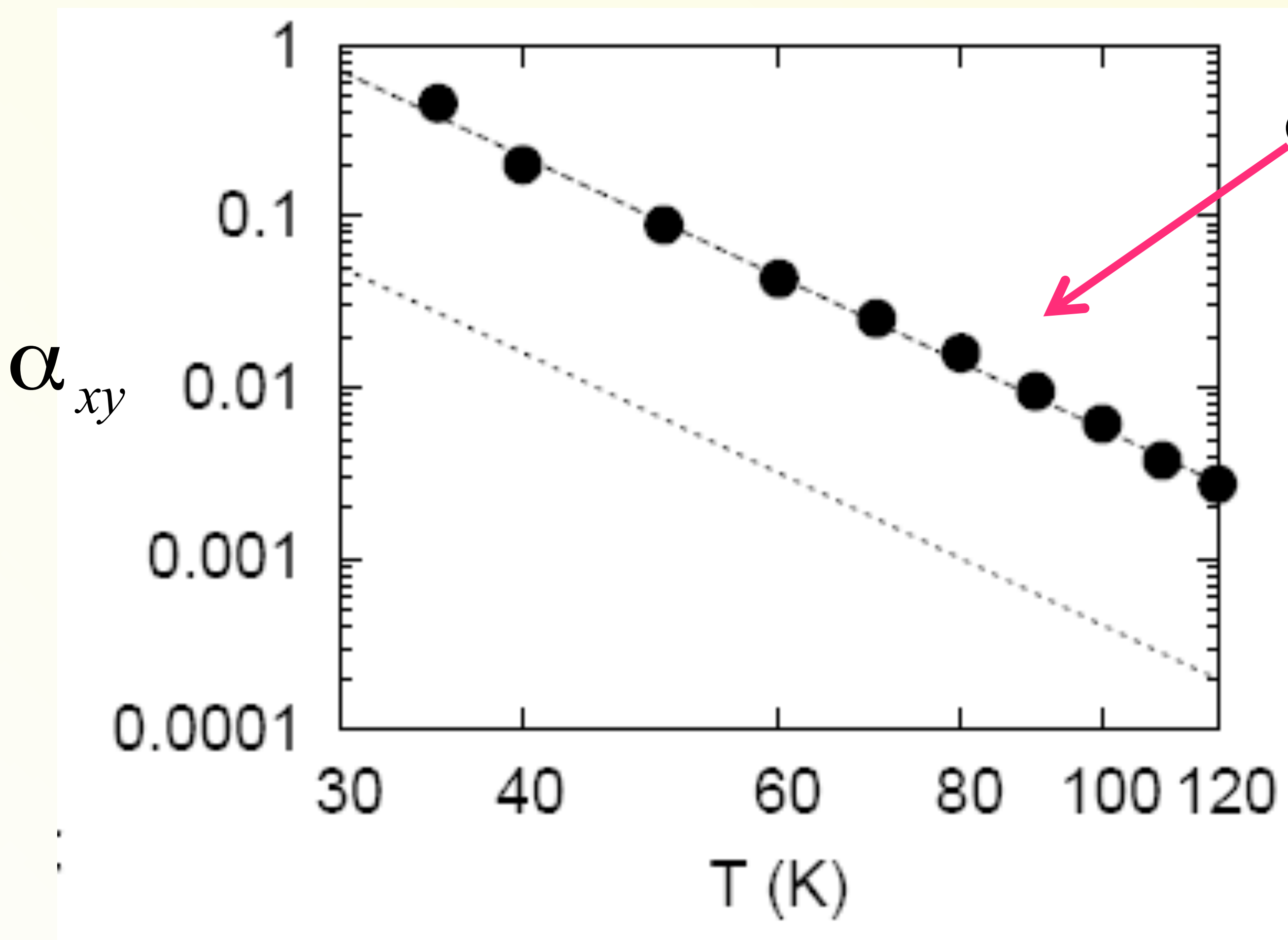
where

$$B = \overline{B}\phi_0/(\hbar v)^2 \quad ; \quad \rho = \overline{\rho}/(\hbar v)^2.$$

S.A. Hartnoll, P.K. Kovtun, M. Müller, and S. Sachdev, *Phys. Rev.* B **76** 144502 (2007)

LSCO Experiments

Measurement of $\alpha_{xy} \approx \sigma_{xx} e_N$



Y. Wang et al., Phys. Rev. B ***73****, 024510 (2006).*

$$\alpha_{xy} \propto \frac{1}{T^4}$$

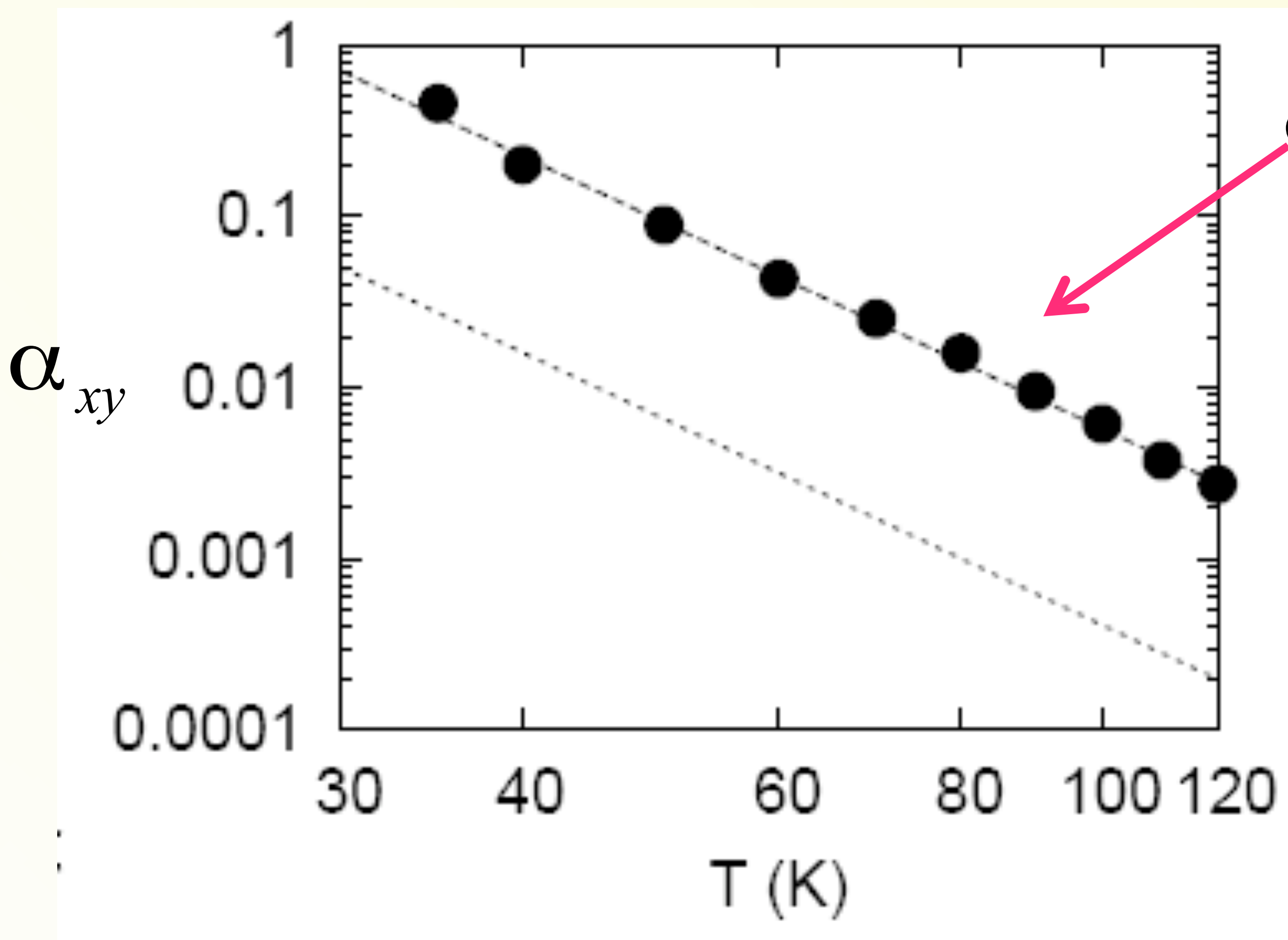
$$\boxed{\alpha_{xy} \propto \frac{BT^2\left(\#\rho^2\tau_{imp} + \#T^3\right)}{T^6 + \#B^2\rho^2\tau_{imp}^2}}$$

(T small)

$$\frac{\alpha_{xy}}{B}(B \to 0) \approx \left(\frac{2ek_B}{h\phi_0}\right)\frac{\Phi_s}{\Phi^2_{\varepsilon+P}}\left(\frac{2\pi\tau_{\rm imp}}{\hbar}\right)^2\frac{\rho^2(\hbar v)^6}{(k_B T)^4}$$

LSCO Experiments

Measurement of $\alpha_{xy} \approx \sigma_{xx} e_N$



Y. Wang et al., Phys. Rev. B ***73****, 024510 (2006).*

$$\alpha_{xy} \propto \frac{1}{T^4}$$

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$\hbar v \approx 47\,\mathrm{meV}\,\mathrm{\mathring{A}}$

$v \approx 2.5\times 10^{-5}c$

$\tau_{imp} \approx 10^{-12}\,\mathrm{s}$

LSCO Experiments

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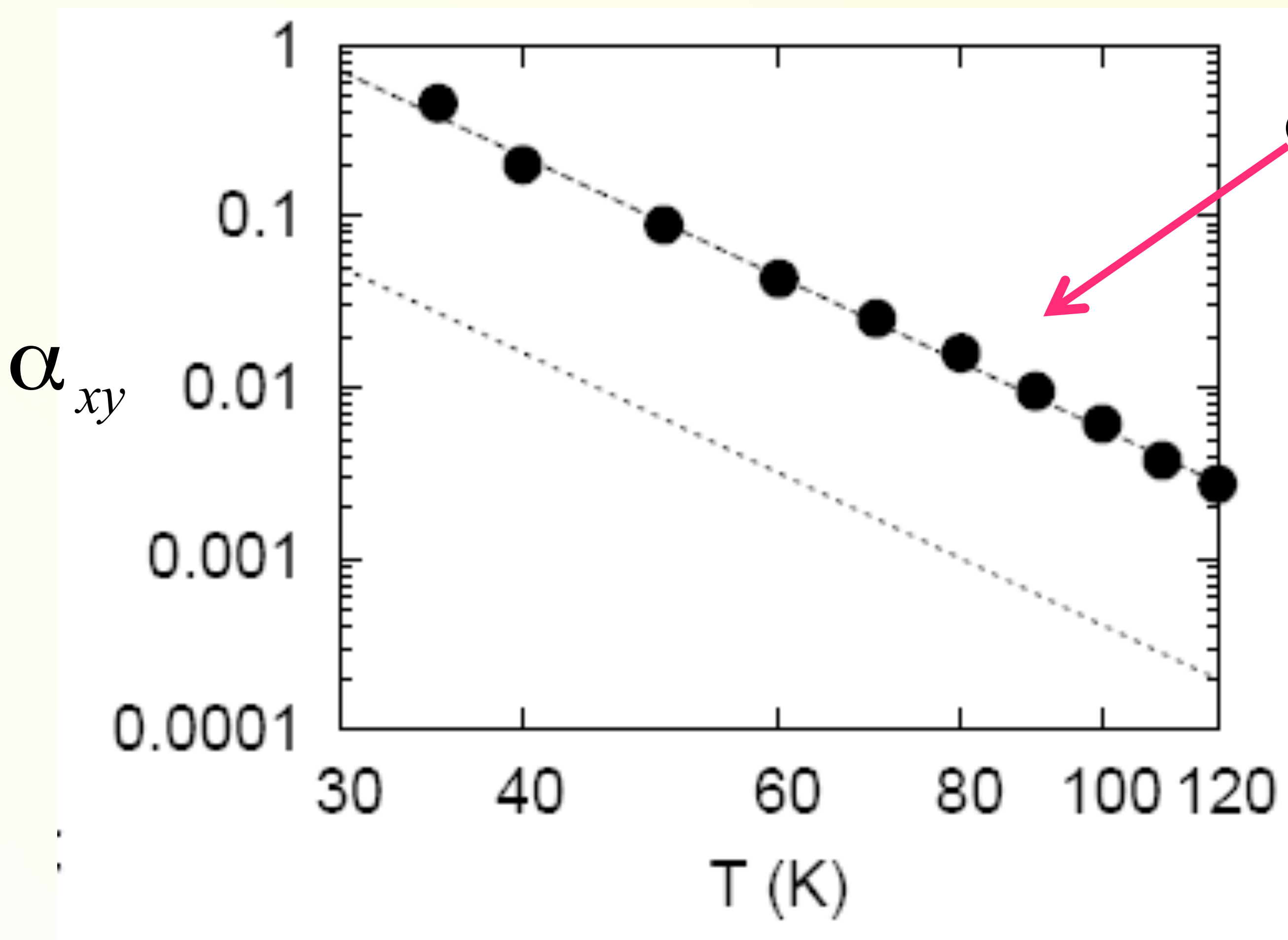

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Y. Wang et al., Phys. Rev. B 73, 024510 (2006).

→ Prediction for ω_c:

$$\omega_c = 6.2\ \text{GHz}\frac{B}{1\ \text{T}}\left(\frac{35\ \text{K}}{T}\right)^3$$

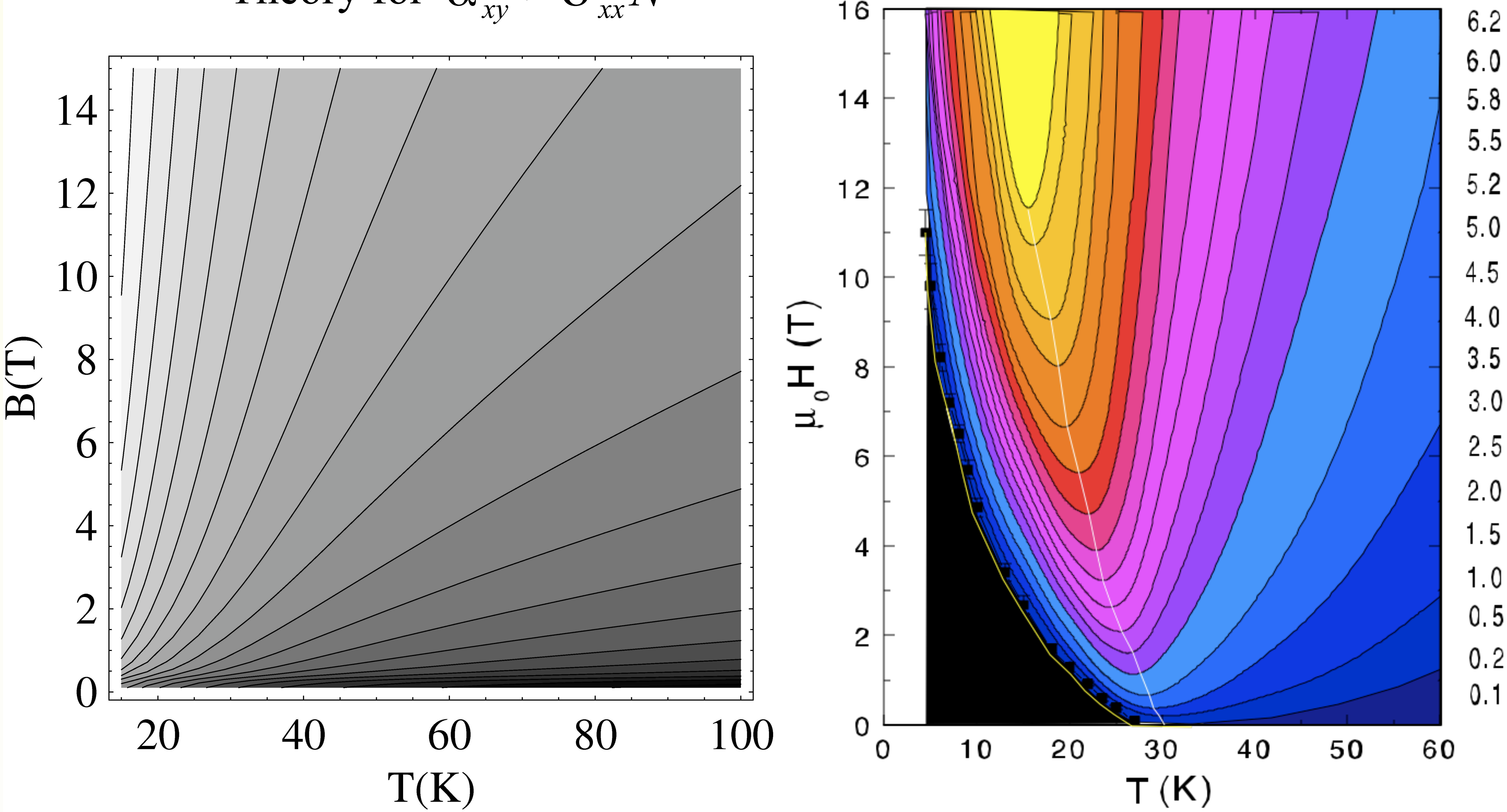
- T-dependent cyclotron frequency!
- 0.035 times smaller than the cyclotron frequency of free electrons (at T=35 K)
- Only observable in ultra-pure samples where $\tau_{imp}^{-1} \leq \omega_c$

LSCO Experiments

B, T-dependence

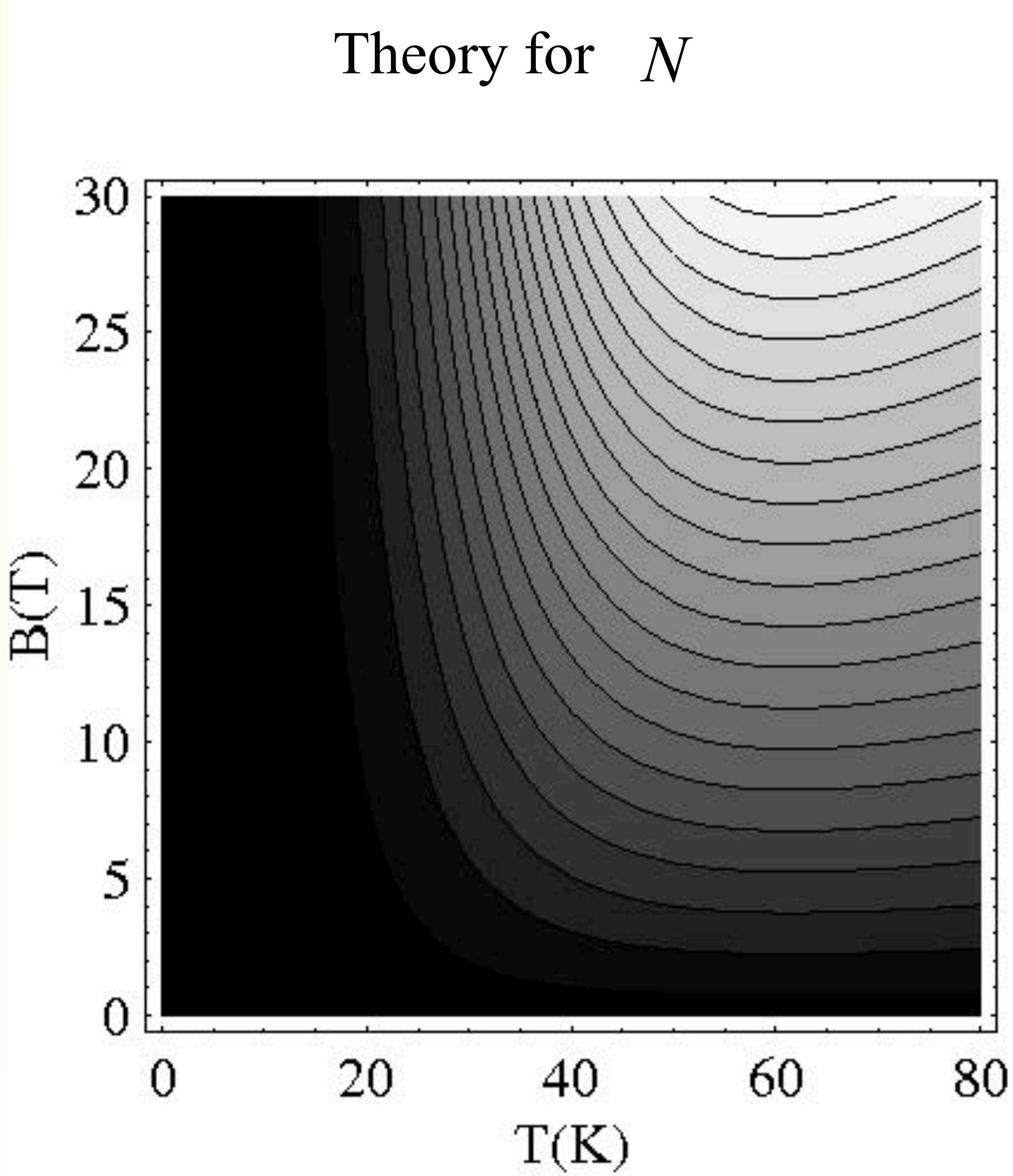
Theory for $\alpha_{xy} \approx \sigma_{xx} N$

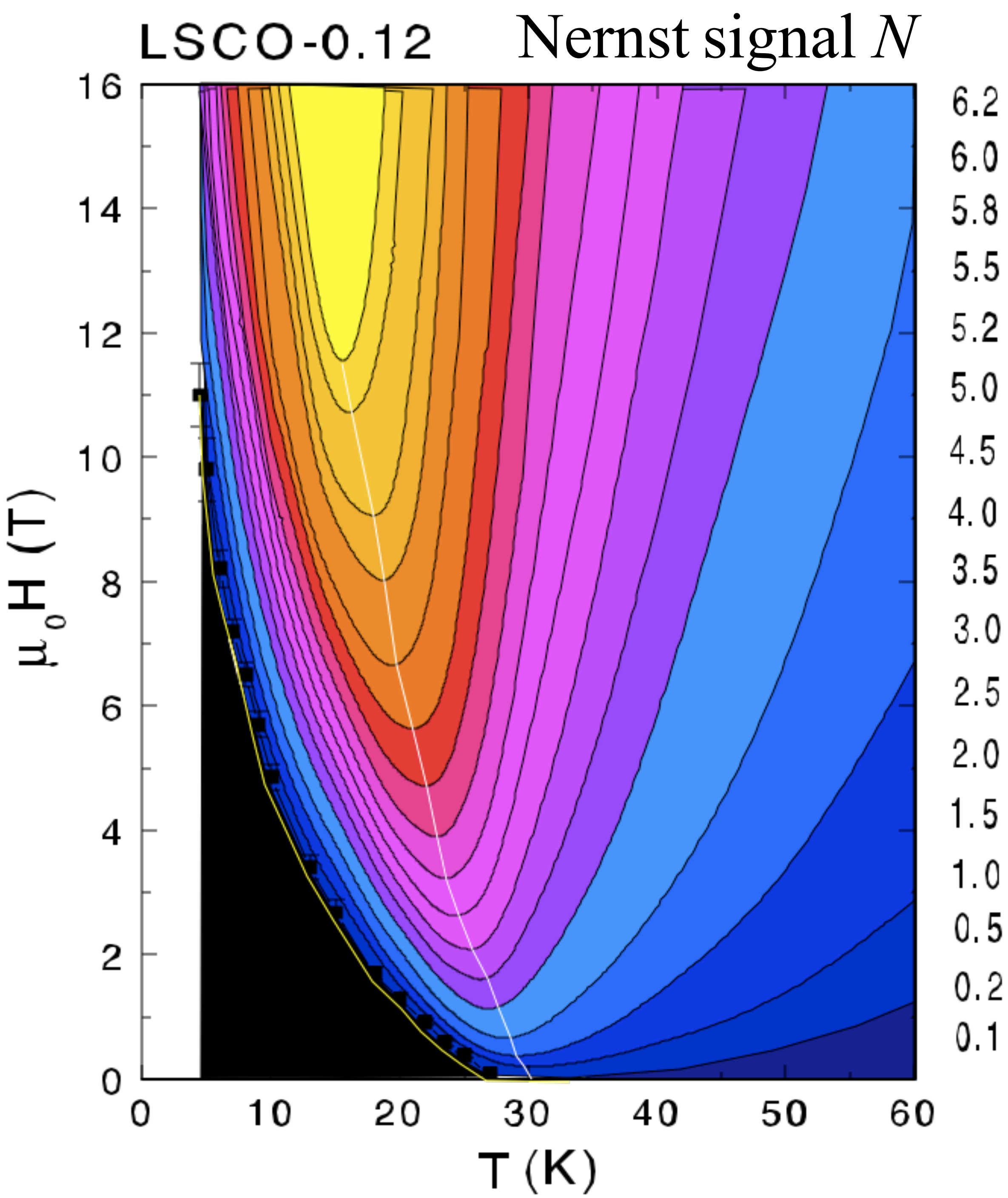
LSCO-0.12 Nernst signal N



Y. Wang, L. Li, and N. P. Ong, Phys. Rev. B ***73****, 024510 (2006).*

LSCO Experiments





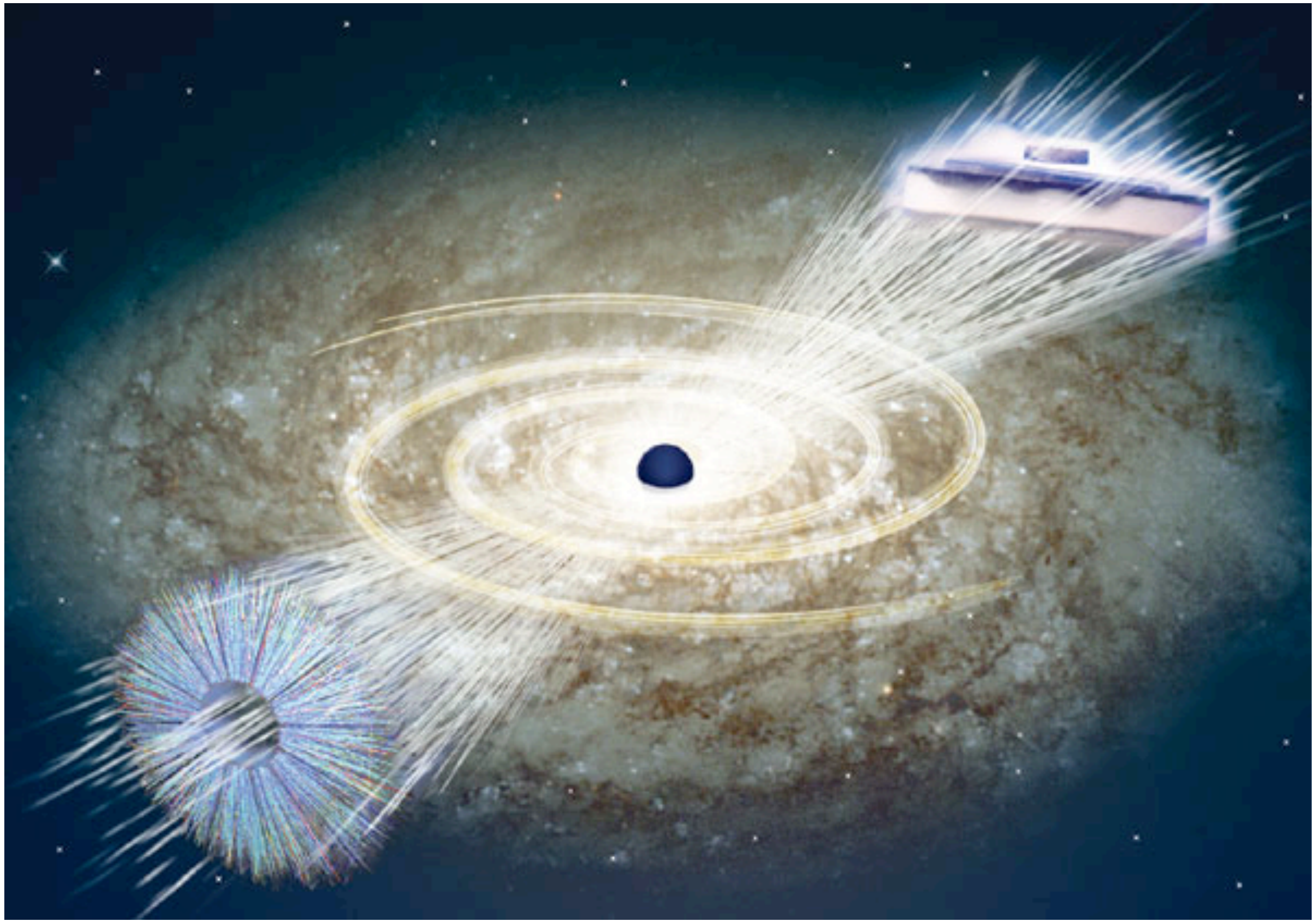
Y. Wang, L. Li, and N. P. Ong, Phys. Rev. B 73, 024510 (2006).

A black hole full of answers

Jan Zaanen

A facet of string theory, the currently favoured route to a 'theory of everything', might help to explain some properties of exotic matter phases — such as some peculiarities of high-temperature superconductors.

NATURE|Vol 448|30 August 2007



Conclusions

- Theory for transport near quantum phase transitions in superfluids and antiferromagnets
- Exact solutions via black hole mapping have yielded first exact results for transport co-efficients in interacting many-body systems, and were valuable in determining general structure of hydrodynamics.
- Theory of Nernst effect near the superfluid-insulator transition, and connection to cuprates.
- Quantum-critical magnetotransport in graphene.