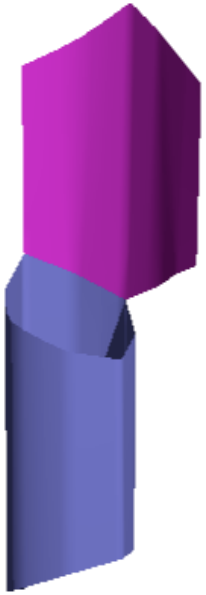
Supersymmetric Extension of The Quantum Hall Effect



Topological Aspects of Solid State Physics

9 June 2008@ISSP

Kazuki Hasebe

Takuma National College of Technology

SUSY QHE with Y. Kimura

NPB (2004), PRL(2005), PRD (2005), PRD (2006), PLA (2008)

SUSY AKLT with D.P. Arovas, X-L. Qi, S-C. Zhang

(in preparation)

Innovative Extensions of the QHE

- **4D Extension of QHE : From U(1) to SU(2)**

  Zhang, Hu (2001)

- **(Intrinsic) Spin Hall Effect**

  Murakami, Nagaosa, Zhang (2003),
  Sinova, Culcer, Niu et. al (2003)

- **Quantum Spin Hall Effect**

  Kane, Mele (2005), Bernevig, Zhang (2006), ...

- **Topological Insulators in Higher Dimensions**

  Moore, Balents (2006), Roy (2006), Fu, Kane (2006),
  Qi, Hughes, Zhang (2008)

Another Possible Extension of the QHE
: SUSY QHE

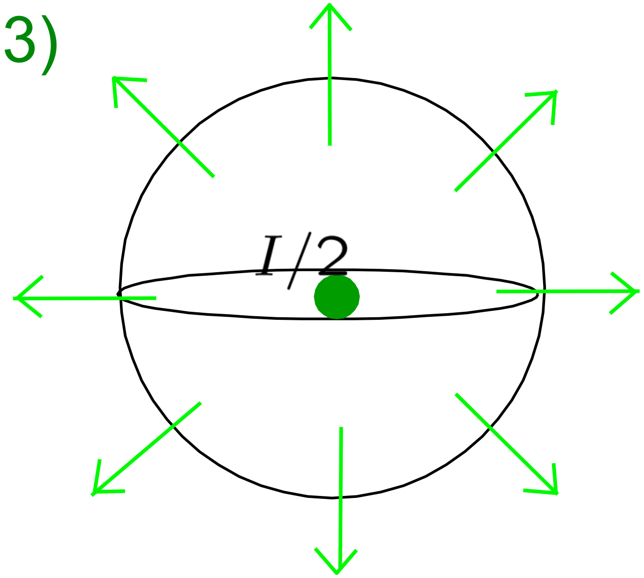
SUSY Quantum Hall Effect

with Yusuke Kimura

A Possible SUSY Extension
of the Haldane's Spherical QHE

The Spherical Quantum Hall Liquid

F.D.M. Haldane (1983)



The Hopf map $S^3 \overset{S^1}{\rightarrow} S^2$

$$\phi \rightarrow x_a = \phi^\dagger \sigma_a \phi$$

$$\phi^\dagger \phi = 1 \qquad x_a^2 = 1$$

Hopf spinor

$$\phi = \begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{\sqrt{2(1+x_3)}} \begin{pmatrix} 1+x_3 \\ x_1 + i x_2 \end{pmatrix} e^{i\chi}$$

$$-i\phi^\dagger d\phi = A_a dx_a$$

$$A_a = \frac{1}{2}\epsilon_{ab3}\frac{x_b}{1+x_3}$$

Laughlin-Haldane wavefunction

$$\Phi^{(m)} = \prod_{i<j}(u_i v_j - v_i u_j)^m$$

: SU(2) singlet

The Math. of QHE is deeply related to the Hopf fibration.

How to include SUSY

The SUSY Hopf map **C. Bartocci, U. Bruzzo, G. Landi (1987)**

$$S^{3|2} \overset{S^1}{\to} S^{2|2}$$

$$\psi \to \begin{cases} x_a = 2\psi^\ddagger l_a \psi & a = 1,2,3 \quad l_a = \frac{1}{2}\begin{pmatrix} \sigma_a & 0 \\ 0 & 0 \end{pmatrix} \\ \theta_\alpha = 2\psi^\ddagger l_\alpha \psi & \alpha = 1,2 \quad l_\alpha = \frac{1}{2}\begin{pmatrix} 0 & \tau_\alpha \\ -(\epsilon\tau_\alpha)^t & 0 \end{pmatrix} \end{cases}$$

$$\psi^\ddagger \psi = 1 \qquad x_a^2 + \epsilon_{\alpha\beta}\theta_\alpha\theta_\beta = 1$$

$$\tau_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \tau_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The Super Hopf spinor

$$\psi = (u, v, \eta)^t \qquad \eta = u\theta_1 + v\theta_2$$

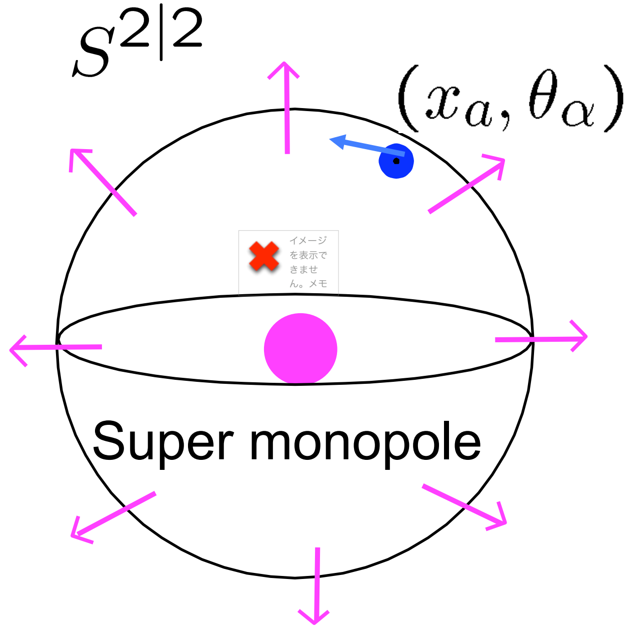
Grassmann-odd

Supermonopole

$$-i\psi^\ddagger d\psi = A_a dx_a + A_\alpha d\theta_\alpha \quad \begin{cases} A_a = \frac{1}{2}\epsilon_{ab3}\frac{x_b}{1+x_3}\left(1 + \frac{2+x_3}{2(1+x_3)}\theta\epsilon\theta\right) \\ A_\alpha = \frac{1}{2}i(\sigma_a\epsilon)_{\alpha\beta}x_a\theta_\beta \end{cases}$$

photon *photino*

Quantum Mechanics on Supersphere



One-particle Hamiltonian

$$H = \frac{1}{2MR^2}(\Lambda_a^2 + \epsilon_{\alpha\beta}\Lambda_\alpha\Lambda_\beta)$$

Covariant SUSY ``angular momenta''

$$\Lambda_a = -i\epsilon_{abc}x_b D_c + \tfrac{1}{2}\theta_\alpha(\sigma_a)_{\alpha\beta}D_\beta,$$

$$\Lambda_\alpha = \tfrac{1}{2}(\epsilon\sigma_a)_{\alpha\beta}x_a D_\beta - \tfrac{1}{2}\theta_\beta(\sigma_a)_{\beta\alpha}D_a$$

$$D_a = \partial_a + iA_a$$

$$D_\alpha = \partial_\alpha + iA_\alpha$$

Conserved SUSY anguler momenta

From particle and supermonopole

$$L_a = \Lambda_a + \frac{I}{2R}x_a,$$ SU(2) angular momenta

$$L_\alpha = \Lambda_\alpha + \frac{I}{2R}\theta_\alpha$$ Supercharges

The Non-anti-commutative Geometry

The OSp(1|2) super-algebra

$$[L_a, L_b] = i\epsilon_{abc}L_c, \quad [L_a, L_\alpha] = \frac{1}{2}(\sigma_a)_{\beta\alpha}L_\beta, \quad \{L_\alpha, L_\beta\} = \frac{1}{2}(\epsilon\sigma_a)_{\alpha\beta}L_a.$$

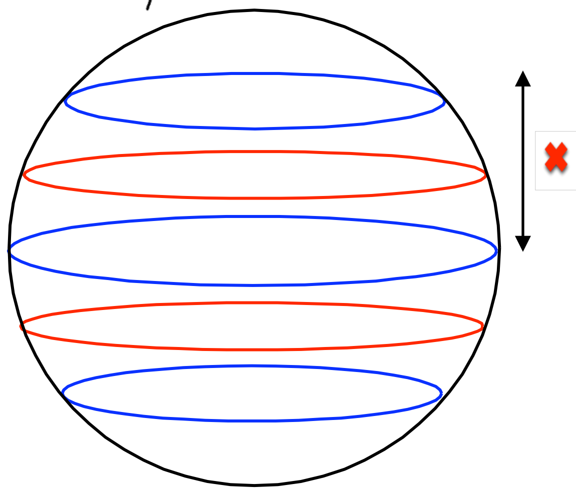
In the lowest Landau level, $x_a \to \alpha L_a, \quad \theta_\alpha \to \alpha L_\alpha.$

$$[X_a, X_b] = i\alpha\epsilon_{abc}X_c,$$

$$[X_a, \Theta_\alpha] = \frac{\alpha}{2}(\sigma_a)_{\beta\alpha}\Theta_\beta,$$

$$\{\Theta_\alpha, \Theta_\beta\} = \frac{\alpha}{2}(\epsilon\sigma_a)_{\alpha\beta}X_a.$$

$\alpha = 2R/I$



Non-anti-commutative Geometry

Fuzzy supersphere H. Grosse & G. Reiter (1998)

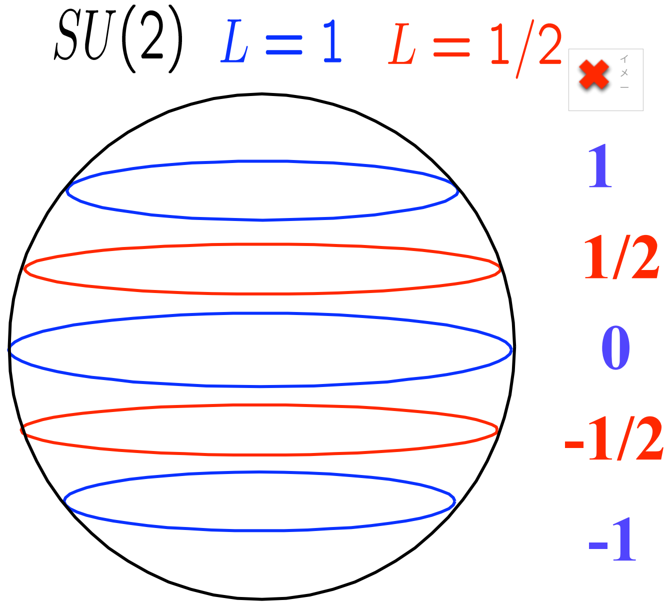
The Math. b.g.d. of the SUSY QHE is NACG.

Supermonopole Harmonics

$OSp(1|2)$ $I/2$ - irrep.

(Ex.) $I/2 = 1$

$\Rightarrow$ $SU(2)$ $I/2 \oplus I/2 - 1/2$



$$u_{m_1,m_2} = \sqrt{\frac{I!}{m_1!m_2!}}\, u^{m_1} v^{m_2},$$

$$\eta_{n_1,n_2} = \sqrt{\frac{I!}{n_1!n_2!}}\, u^{n_1} v^{n_2} \eta$$

(No complex conjugate variables)

$m_1 + m_2 = I$ $\qquad$ $n_1 + n_2 = I - 1$

$$d(I) = \underbrace{(I+1)}_{\text{Bosonic d.o.f.}} + \underbrace{(I)}_{\text{Fermionic d.o.f.}} = 2I + 1 \propto 2I$$

Bosonic d.o.f. Fermionic d.o.f.

The SUSY system is (almost) doubly degenerate due to the existence of the fermionic d.o.f.

Supersymmetric Laughlin Wavefunction

The SUSY Laughlin-Haldane wavefunction

$$\Psi^{(m)} = \prod_{i<j}(u_i v_j - v_i u_j - \eta_i \eta_j)^m$$

: OSp(1|2) singlet

The SUSY Laughlin wavefunction

$$\Psi^{(m)} = \prod_{i<j}(z_i - z_j + \theta_i \theta_j)^m e^{-\sum_i (z_i z_i^* + \theta_i \theta_i^*)}$$

Analogy to BCS state

Pairing-operator

$$|BCS> = \prod_k (1 + g_k c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow})|0> = \exp(\sum_k g_k c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow})|0>$$

$$\Psi^{(m)} = \exp(m \sum_{i<j} \frac{\theta_i \theta_j}{z_i - z_j}) \cdot \Phi^{(m)}$$

The original Laughlin func.

Interpretation of Fermionic Variables

Planar Limit

$$[X, Y] = i\ell_B^2, \quad \{\Theta_1, \Theta_2\} = \ell_B^2.$$

Raising, Lowering operators

$$(X + iY)/\sqrt{2}\ell_B \to L^+ \qquad \Theta_1/\ell_B \to \sigma^+ = \sigma_x + i\sigma_y$$

$$(X - iY)/\sqrt{2}\ell_B \to L^- \qquad \Theta_2/\ell_B \to \sigma^- = \sigma_x - i\sigma_y$$

$$[L^+, L^-] = 1, \qquad \{\sigma^+, \sigma^-\} = 1.$$

Orbital angular momenta

$z = x + iy$ → leftward-rotation

$z^* = x - iy$ → rightward-rotation

Spin angular momenta

$\theta = \theta_1$ → up-spin

$\theta^* = \theta_2$ → down-spin

The fermionic variables may be interpreted as spin d.o.f.

Interpretation of the Pairing Operator

The SUSY Laughlin state as a ``superfield''

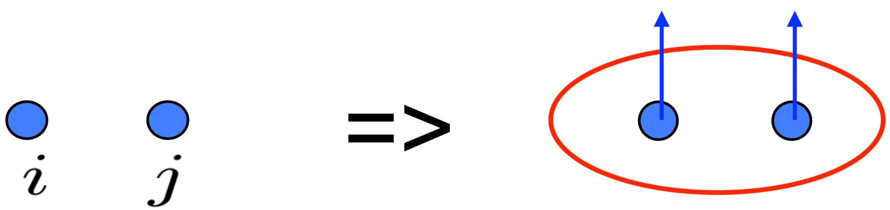
$$\Psi^{(m)} = \exp\left(m \sum_{i<j} \frac{\theta_i \theta_j}{z_i - z_j}\right) \cdot \Phi^{(m)}$$

The pairing operator

$$\sum_{i<j} \frac{\theta_i \theta_j}{z_i - z_j}$$

Adding 1/2-spin d.o.f. to (*i, j*) particles

p-wave pairing of (*i, j*) particles



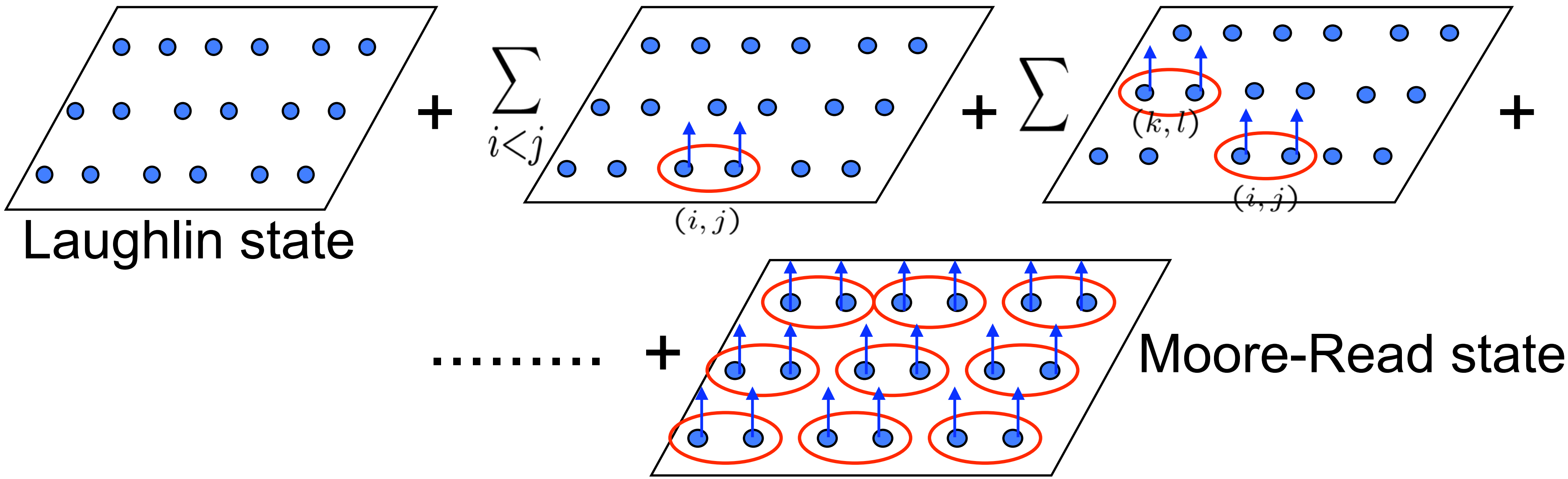
p-wave pairing
with polarized spins

Expansion of SUSY Laughlin Function

$$\Psi^{(m)} = \Phi^{(m)} + m\sum_{i<j}\frac{\theta_i\theta_j}{z_i-z_j}\Phi^{(m)} + \frac{m^2}{2}\left(\sum_{i<j}\frac{\theta_i\theta_j}{z_i-z_j}\right)^2\Phi^{(m)} +$$

$$\cdots + \frac{m^{\frac{N}{2}}}{(N/2)!}\theta_1\theta_2\cdots\theta_N\cdot Pf\left(\frac{1}{z_i-z_j}\right)\cdot\Phi^{(m)}$$

All spins polarized

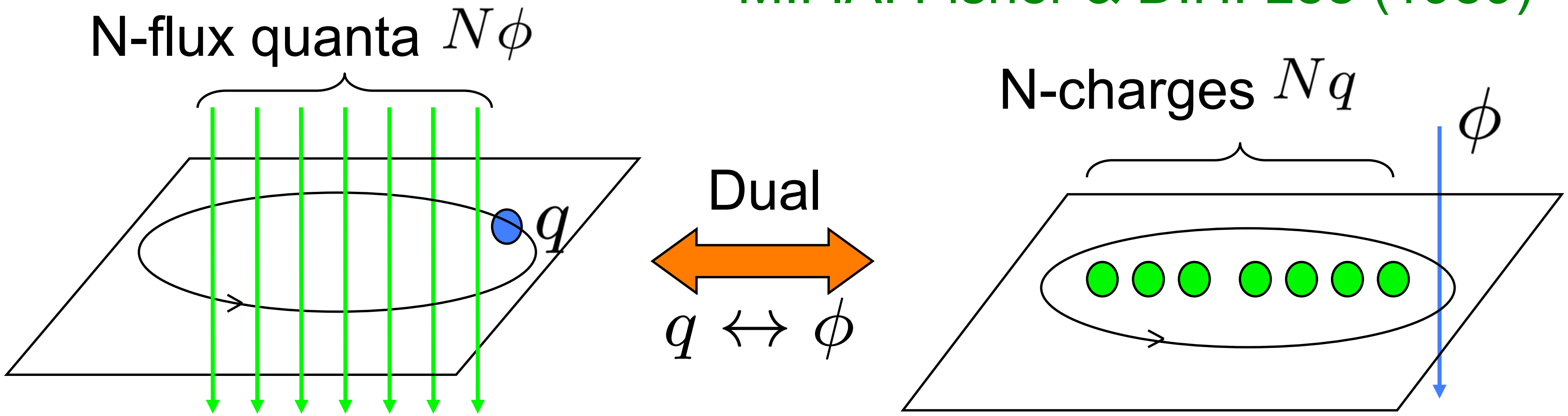


The SUSY unifies the two novel QH liquids.

	Haldane's set-up	SUSY set-up
Base Manifold	$S^2 = SU(2)/U(1)$	$S^{2\|2} = OSp(1\|2)/U(1)$
Hopf Map	$S^3 \to S^2$	$S^{3\|2} \to S^{2\|2}$
Monopole	Dirac Monopole	Supermonopole
N.C. Manifold	Fuzzy sphere	Fuzzy supersphere
Ground-state	SU(2) inv. Laughlin func.	OSp(1|2) inv. Laughlin func.
Analogy to Superfluidity	Condensation of composite bosons	p-wave pairings on the original Laughlin state

Charge-Flux Duality in 3(=2+1) D

M.P.A. Fisher & D.H. Lee (1989)



$$e^{iq\oint A} = e^{iqN\phi}$$

Currents (Charge) ⟷ Field strengths (Flux)

J_a : 3 | F_{ab} : 3

$a = 1, 2, 3$ | $(F_a = \epsilon_{abc} F_{bc})$

$\partial_a J_a = 0$ | $\partial_a F_a = 0$

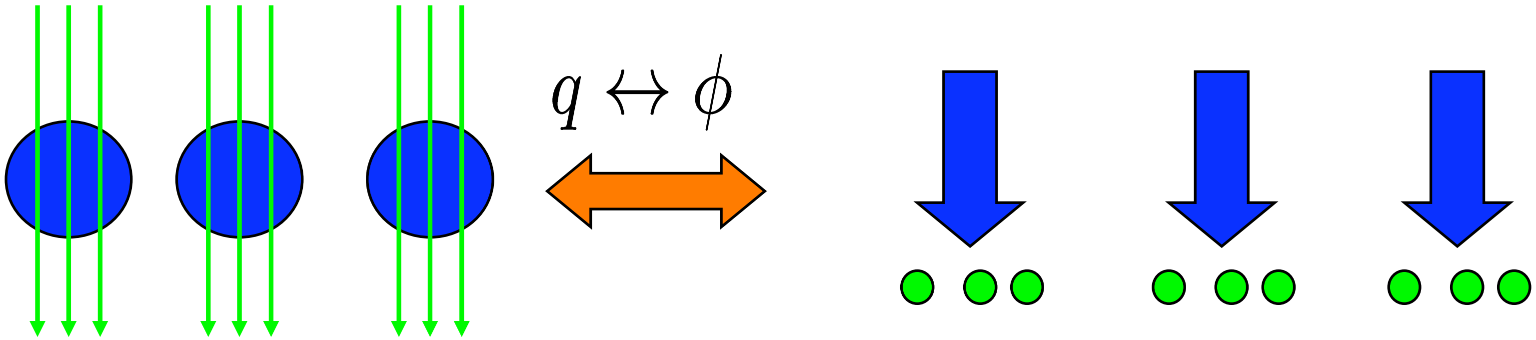
Charge-flux duality exists for the specialty of 3D.

Chern-Simons Theory and Duality in QHE

S.C. Zhang, T.H. Hansson & S. Kivelson (1989),
D.H. Lee & C.L. Kane (1989), D.H. Lee & S.C. Zhang (1991)

$\nu = 1/3$ | $\tilde{\nu} = 3$

CS flux | Dual CS flux



$$L_{CS} = \nu A_a F_a = \nu \epsilon_{abc} A_a \partial_b A_c \quad \overset{S}{\Longleftrightarrow} \quad L_{dualCS} = 1/\nu \tilde{A}_a \tilde{F}_a = 1/\nu \epsilon_{abc} \tilde{A}_a \partial_b \tilde{A}_c$$

Theory of particles | Theory of vortices

Dual description is crucial for the study of topological objects.

The SUSY Chern-Simons Theory

The Super-Currents

$$(J_a, J_\alpha) \quad : 3+2=5$$

The Super-Field Strengths

$$F_{ab},\quad F_{a\alpha},\quad F_{\alpha\beta}$$

: 3+6+3=12

Specialty in 3|2 →

$$F_a = \frac{1}{2}\epsilon_{abc}F_{bc} + i\frac{1}{4}(\epsilon\sigma_a)_{\alpha\beta}F_{\alpha\beta}$$

$$F_\alpha = -i\frac{1}{2}(\epsilon\sigma_a)_{\alpha\beta}F_{a\beta}$$

$$(J_a, J_\alpha) \overset{\text{dual}}{\Longleftrightarrow} (F_a, F_\alpha)$$

$$\partial_a J_a + \partial_\alpha J_\alpha = 0 \qquad \partial_a F_a + \partial_\alpha F_\alpha = 0$$

The SUSY Chern-Simons Lagrangian

$$\begin{aligned} L_{sCS} &= A_a F_a + A_\alpha F_\alpha \\ &= \epsilon_{abc}A_a\partial_b A_c - i(\epsilon\sigma_a)_{\alpha\beta}A_\alpha\partial_a A_\beta + 2i(\epsilon\sigma_a)_{\alpha\beta}A_\alpha\partial_\beta A_a \end{aligned}$$

Properties of the SUSY Chern-Simons Lagrangian

1.OSp(1|2) global SUSY

2. U(1) gauge invariance up to total derivatives

3. Coupled to Maxwell Lagrangian,
both A_a, A_α acquire topological masses.

4. SUSY Chern-Simons-Landau-Ginzburg Theory

$$L_{CSLG} = J_a A_a + J_\alpha A_\alpha + \frac{\nu}{4\pi}(F_a A_a + F_\alpha A_\alpha) + \cdots$$

Dual trans.

$$L_{dualCSLG} = \tilde{J}_a \tilde{A}_a + \tilde{J}_\alpha \tilde{A}_\alpha + \frac{1}{4\pi\nu}(\tilde{F}_a \tilde{A}_a + \tilde{F}_\alpha \tilde{A}_\alpha) + \cdots$$

The SUSY CS theory inherits the properties of the original CS theory in the SUSY sense.

THE SUSY AKLT MODEL

with Daniel P. Arovas,
Xiaoliang Qi,
Shoucheng Zhang

From SUSY QHE to SUSY Valence Bond Solid

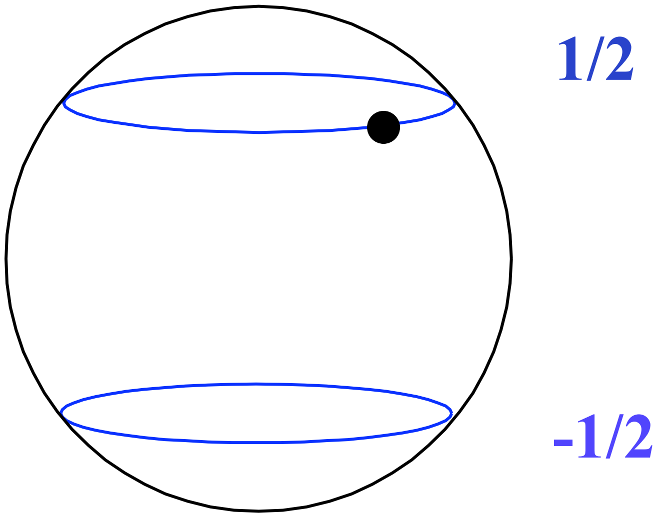
Analogies between LLL and Spin States

LLL states

(Monopole Harmonics)

$$L = I/2 = 1/2$$

External (Real) space

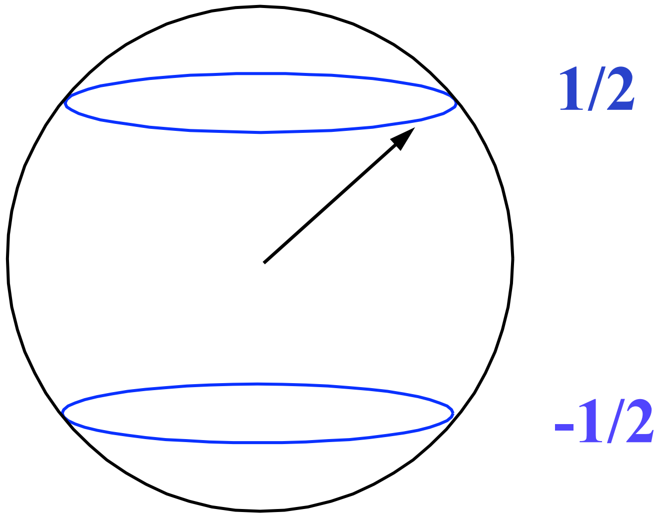


Fuzzy sphere

SU(2) spin states

$$S = 1/2$$

Internal Spin space



Bloch sphere

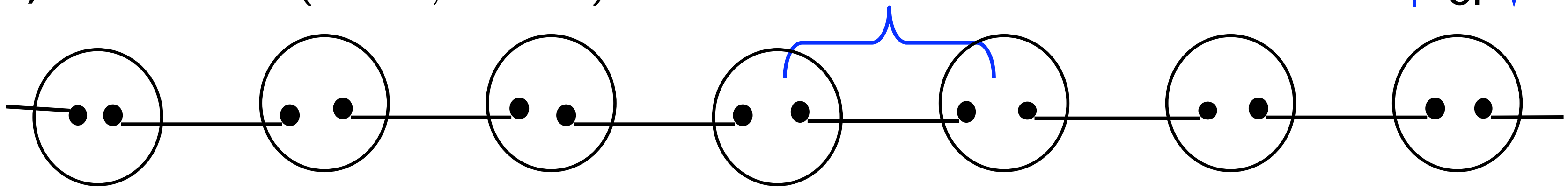
Analogies in Many-body States

The AKLT state Affleck, Kennedy, Lieb, Tasaki (1987)

$$|AKLT> = \prod_{<ij>}^{z} (\underbrace{a_i^\dagger b_j^\dagger - a_j^\dagger b_i^\dagger}_{\text{SU(2) singlet}})^M |0>$$

Spin $S(i) = 1/2(a^\dagger a + b^\dagger b)_i \Rightarrow S = zM/2$

(Ex.) $S = 1 \quad (z = 2, M = 1)$ Valence bond ● = ↑ or ↓



The Coherent State Representation $(a, b) \rightarrow (u, v)$

Arovas, Auerbach, Haldane (1988)

$$\Phi_{AKLT} = \prod_{<ij>}^{z} (u_i v_j - u_j v_i)^M$$

(Analogous to the Laughlin-Haldane func.)

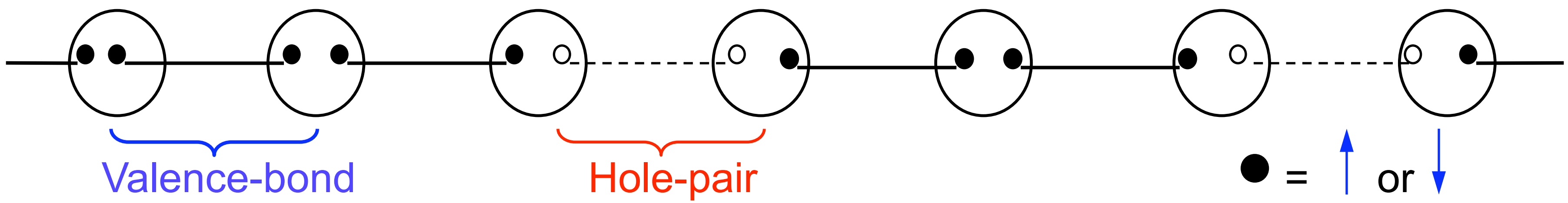
The SUSY AKLT State

$$|SUSY\,AKLT> = \prod_{<ij>}^{z} (\underbrace{a_i^\dagger b_j^\dagger - a_j^\dagger b_i^\dagger}_{\text{Valence-bond}} - \underbrace{f_i^\dagger f_j^\dagger}_{\text{Hole-pair}})^M |0>$$

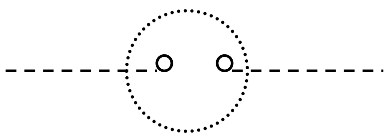
The total particle number $(a^\dagger a + b^\dagger b + f^\dagger f)_i$: constant zM

Spin $S(i) = 1/2(a^\dagger a + b^\dagger b)_i$ ⇨ $zM/2$ or $zM/2 - 1/2$

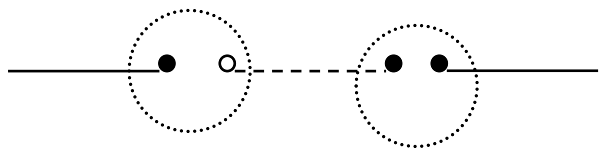
(Ex.) $S = 1 \; or \; 1/2 \; (z=2, \; M=1)$



Non-existing Terms



$f_i^{\dagger 2} = 0$



Properties of the SUSY AKLT Chain

The SUSY AKLT chain in the (spin-hole) coherent state repr.

$$\Psi_{AKLT} = \prod_i (u_i v_{i+1} - u_{i+1} v_i + r \eta_i \eta_{i+1})$$

$(r = -1)$

Simply rewritten as

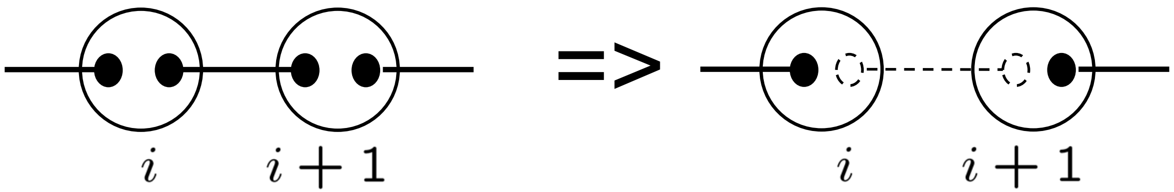
$$\Psi_{AKLT} = \exp\left(r \sum_i \frac{\eta_i \eta_{i+1}}{u_i v_{i+1} - u_{i+1} v_i}\right) \cdot \Phi_{AKLT}$$

Bond-breaking operator

$$\frac{\eta_i \eta_{i+1}}{u_i v_{i+1} - u_{i+1} v_i}$$

Adding a hole pair ($\eta_i \eta_{i+1}$)

Breaking the valence bond ($u_i v_{i+1} - u_{i+1} v_i$)

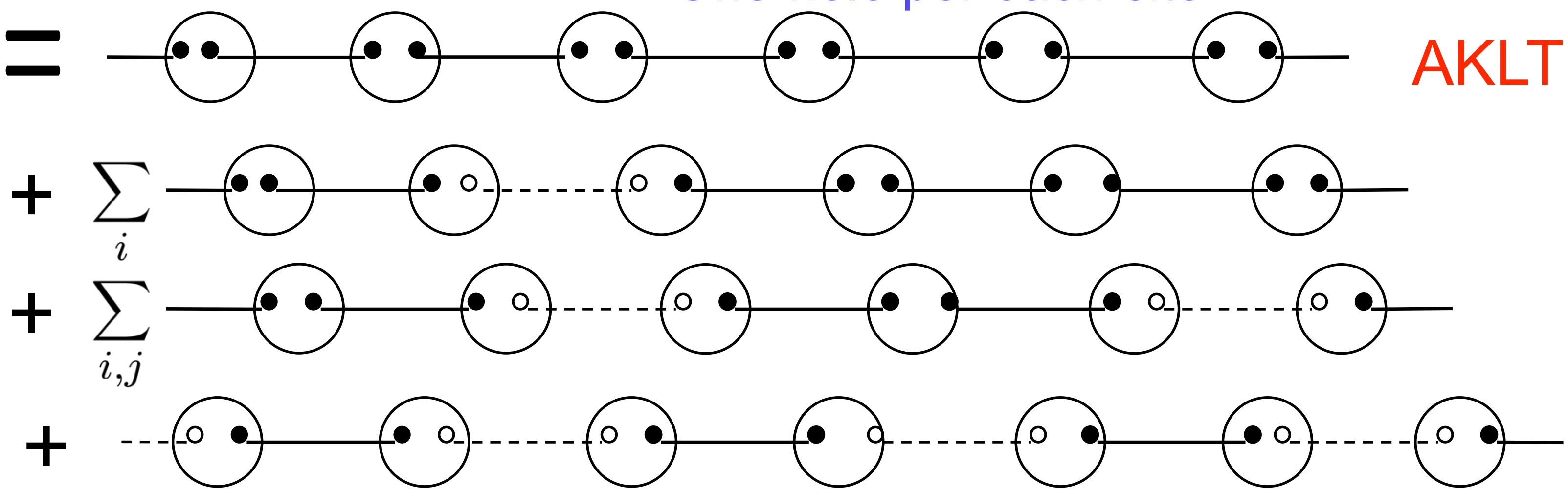


Breaking the bond,
Creating a hole-pair

Expansion of the SUSY AKLT Chain

$$\Psi_{AKLT} = \Phi_{AKLT} + r\sum_i \frac{\eta_i\eta_{i+1}}{u_i v_{i+1} - v_i u_{i+1}}\Phi_{AKLT} + \frac{r^2}{2}\left(\sum_i \frac{\eta_i\eta_{i+1}}{u_i v_{i+1} - v_i u_{i+1}}\right)^2\Phi_{AKLT} + \cdots + r^N \eta_1\eta_2\cdots\eta_N \cdot \left(\Phi_{MJ}^{(even)} - \Phi_{MJ}^{(odd)}\right)$$

One-hole per each-site



AKLT State

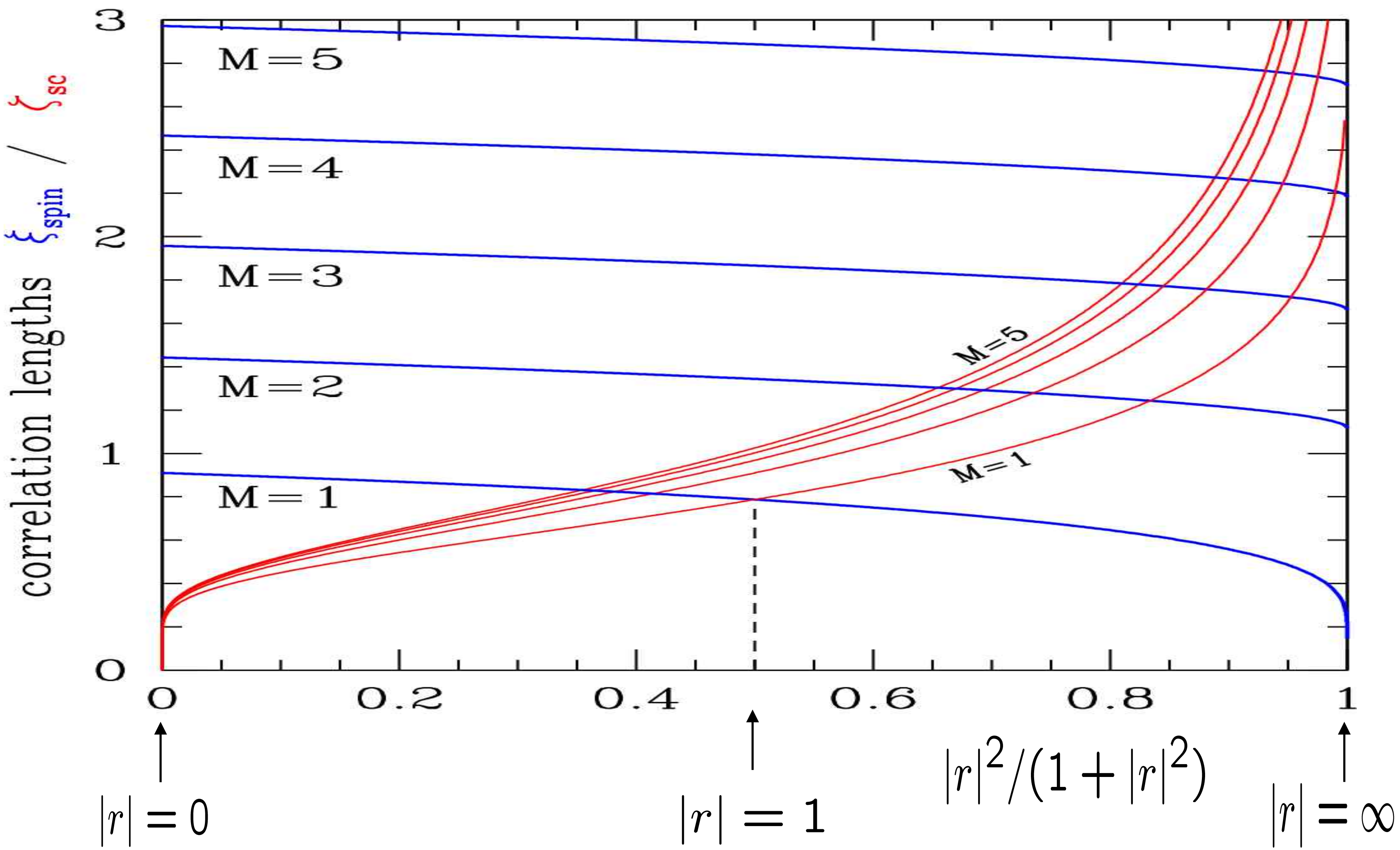
$$\Phi_{MJ}^{(even\ or\ odd)} = \prod_{i:\ even\ or\ odd} (u_i v_{i+1} - u_{i+1} v_i)$$

Majumdar Gosh Dimer State

The SUSY AKLT realizes the two VB states in limits.

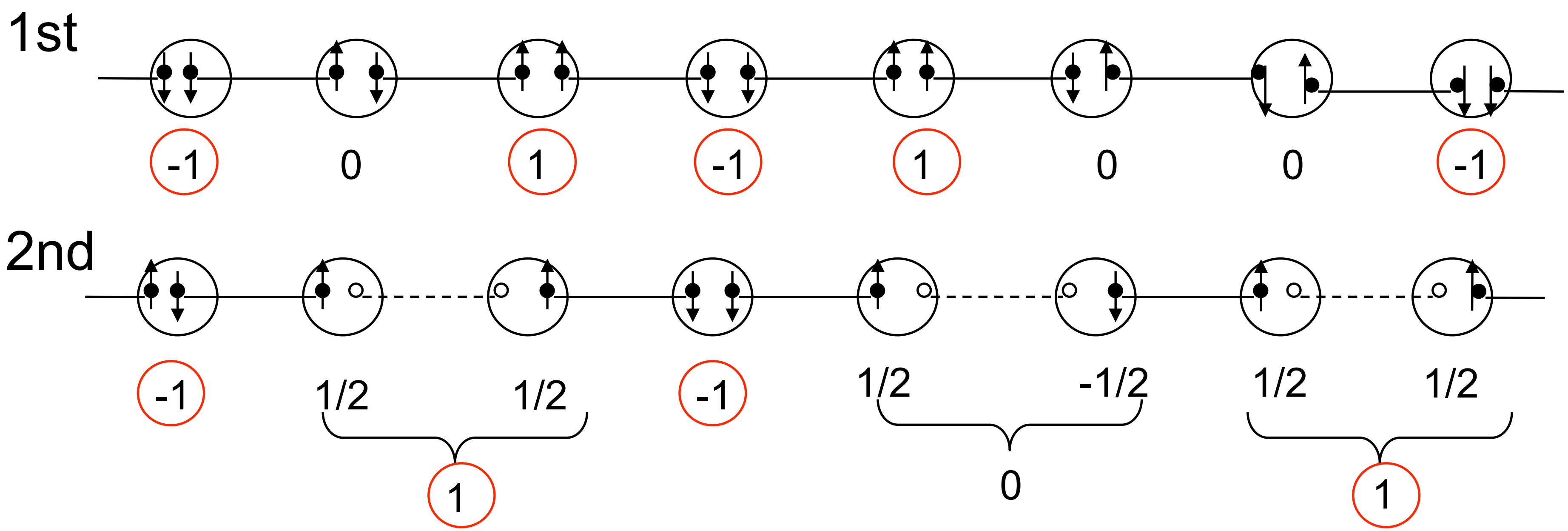
r-dependence of the correlation lengths

$$< S_a(i) S_a(j) > \approx e^{-|i-j|/\xi_{spin}} \qquad < (a_i b_j - b_i a_j) f_i^\dagger f_j^\dagger > \approx e^{-|i-j|/\zeta_{sc}}$$

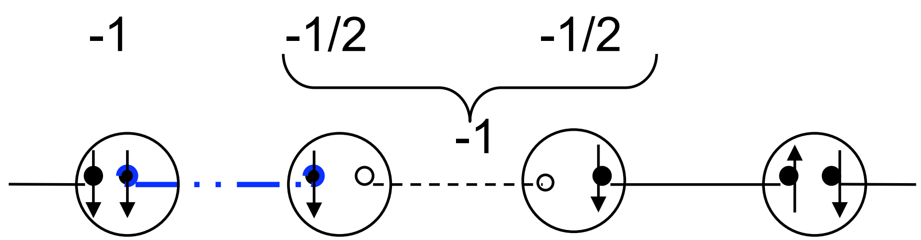


Hidden Order in the SUSY AKLT State

Possible states in the SUSY AKLT state



Non-existing States



The SUSY AKLT Shows a Generalized Hidden Order.

Summary

We developed a SUSY formulation of the quantum Hall effect based on the SUSY Hopf map and the OSp(1|2) super group.

Main Results

- Emergence of Non-anti-commutative Geometry
- A Unified description of Laughlin and Moore-Read states
- Construction of the SUSY Chern-Simons theory
- Application to the VB state

Issues to be Explored

- Topological order? Topological algebra?
- Edge states?
- Relation to Integrable systems?
- Many SUSY? Higher Dimensions?

etc, etc.