Goldstone-mode Instability in a Spinor Bose-Einstein Condensate

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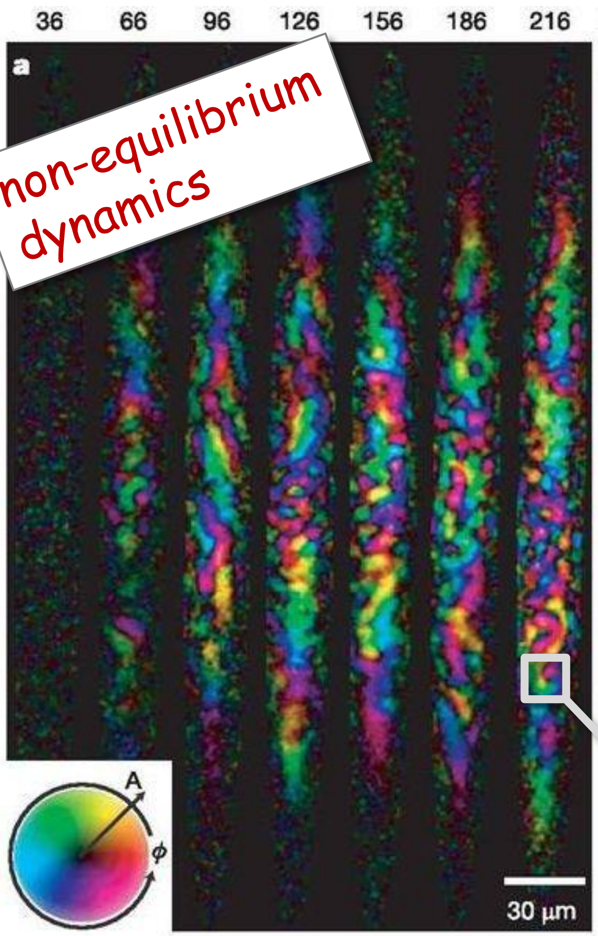
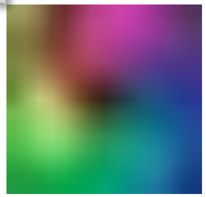
Cold-atomic Systems

non-equilibrium dynamics

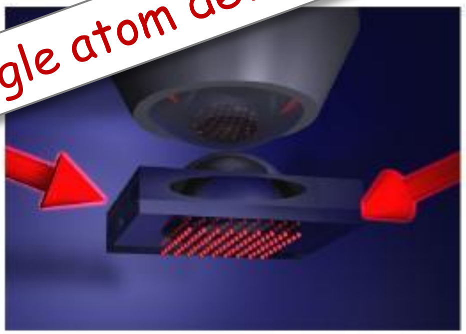


Sadler et al., Nature (2006)

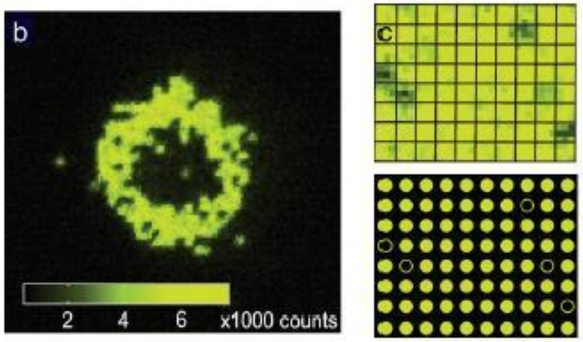


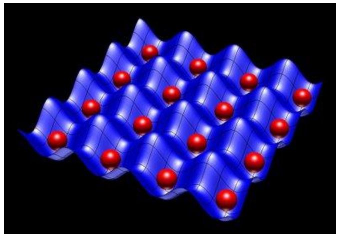
spin vortex

single atom detection



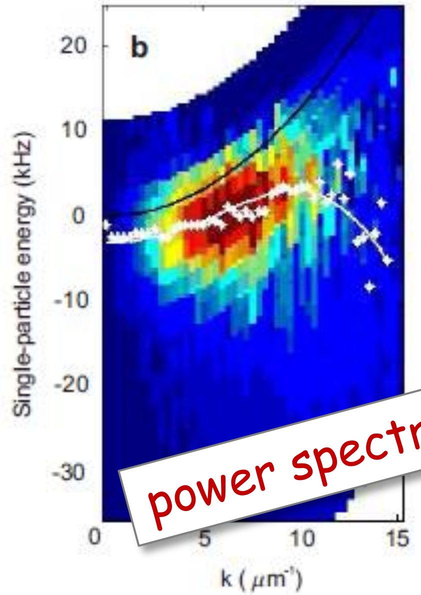
Bakr et al., Science (2010)





Optical lattice
triangle, hexagonal, kagome

Stewart et al., Nature (2008)



power spectrum

Outline

- Introduction: Spinor BEC
  - Spontaneous symmetry breaking and topological excitations
  - Spin-singlet pair condensation

- Bogoliubov analysis
  - Dynamical instability
  - Quantum depletion

- Number conserving Bogoliubov analysis
  - Instability of zero energy mode

Internal Degrees of Freedom

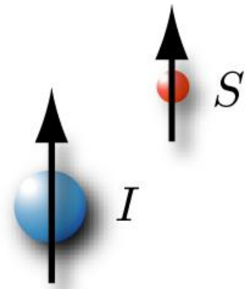
hyperfine spin

$$\boldsymbol{F} = \boldsymbol{S} + \boldsymbol{L} + \boldsymbol{I}$$

$\boldsymbol{S}$: electron spin

$\boldsymbol{L}$: electron orbital

$\boldsymbol{I}$: nuclear spin



Boson

^{87}Rb, ^{23}Na, ^{7}Li, ^{41}K	F=1, 2
^{85}Rb	F=2, 3
^{133}Cs	F=3, 4
^{52}Cr	S=3, I=0
^{164}Dy	S=2, L=6, I=0
^{168}Er	S=1, L=5, I=0

Fermion

^{6}Li	F=1/2,3/2
^{40}K	F=7/2,9/2
^{171}Yb	S=0, I=1/2
^{173}Yb	S=0, I=5/2

Direct Observation of the Order Parameter

Mean-field state (Hartree)

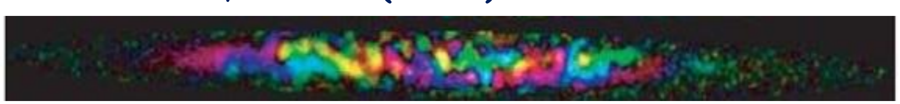
$$\Psi_N(\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_N) = \prod_{i=1}^{N} \varphi(\boldsymbol{r}_i)$$

$$\varphi(\boldsymbol{r}) = \frac{1}{\sqrt{N}} \sum_{m=-F}^{F} \psi_m(\boldsymbol{r}) \chi_m$$

macroscopic wave function
= order parameter

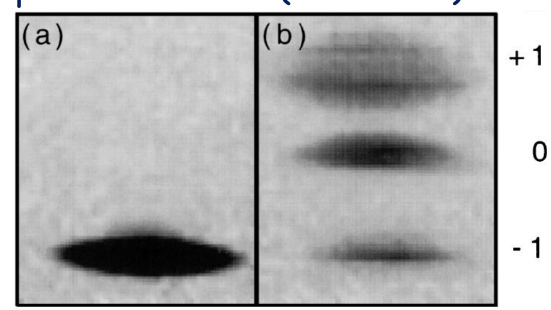
Phase-contrast imaging
(phase difference between spin components)

Sadler et al., Nature (2006)

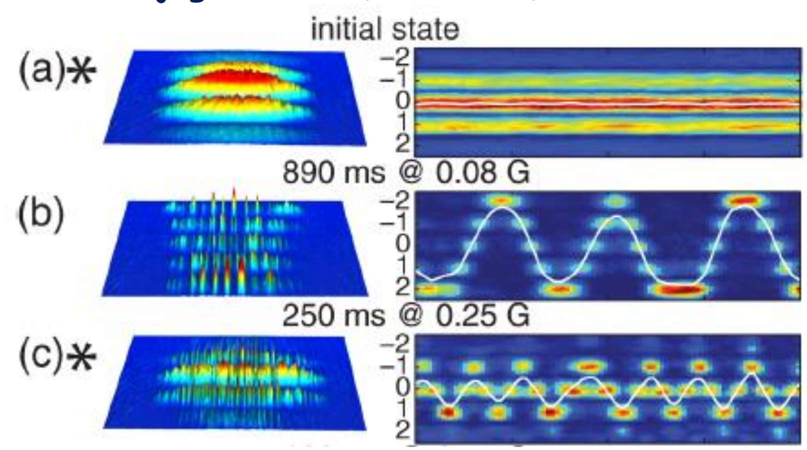


Stern-Gerlach experiment
(density profile)

F=1: Stamper-Kurn et al. (PRL 1998)



F=2: Kronjager et al. (PRL 2010)



Mean-Field Energy Functional : spin-1

$$E[\psi] = \int d\boldsymbol{r} \sum_m \left[-\frac{\hbar^2}{2M} \psi_m^*(\boldsymbol{r}) \nabla^2 \psi_m(\boldsymbol{r}) \right]$$

$$+ \frac{1}{2} \iint d\boldsymbol{r} d\boldsymbol{r}' \sum_{mnm'n'} \psi_m^*(\boldsymbol{r}) \psi_{m'}^*(\boldsymbol{r}') V_{mnm'n'}(\boldsymbol{r} - \boldsymbol{r}') \psi_{n'}(\boldsymbol{r}') \psi_n(\boldsymbol{r})$$

where $V_{mm'n'n}(\boldsymbol{r}, \boldsymbol{r}') = \delta(\boldsymbol{r} - \boldsymbol{r}') \left[\underbrace{c_0 \delta_{mm'} \delta_{nn'}}_{\text{spin-independent}} + \underbrace{c_1 \boldsymbol{F}_{mm'} \cdot \boldsymbol{F}_{nn'}}_{\text{spin-exchange}} \right] \qquad c_0 \gg |c_1|$

$\boldsymbol{F}$: Spin 1 matrix

$$F_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad F_y = \frac{1}{\sqrt{2}i} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad F_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Symmetry of the Hamiltonian

$$G = U(1) \times SO(3) = \{ e^{i\phi} e^{-iF_z\alpha} e^{-iF_y\beta} e^{-iF_z\gamma} \}$$

Ground state $\quad |\langle \boldsymbol{F} \rangle| = \begin{cases} n & \text{ferromagnetic } (c1<0) \\ 0 & \text{polar } (c1>0) \end{cases}$

Polar BEC

e.g. ^{23}Na BEC

- Order parameter $\begin{pmatrix} \psi_1 \\ \psi_0 \\ \psi_{-1} \end{pmatrix} = \sqrt{n} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

visualize the symmetry

$$\Psi = \sum_{m=-f}^{f} Y_{fm}(\theta, \phi)\psi_m$$

- Symmetry of the Polar state

$$H = U(1)_\phi \times (Z_2)_{\phi,\boldsymbol{F}} = (D_\infty)_{\phi,\boldsymbol{F}}$$

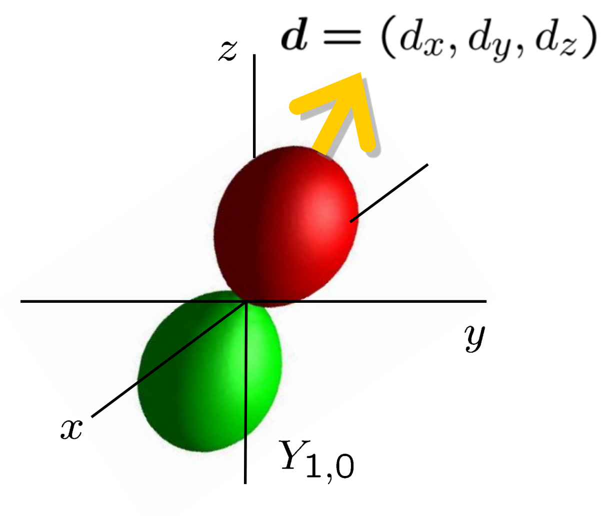
- Order parameter manifold $M = G/H$

general order parameter for the polar state

$$\begin{pmatrix} \psi_1 \\ \psi_0 \\ \psi_{-1} \end{pmatrix} = \sqrt{n}\,\frac{e^{i\theta}}{\sqrt{2}} \begin{pmatrix} -d_x + id_y \\ \sqrt{2}d_z \\ d_x + id_y \end{pmatrix}$$

θ : overall phase

$\boldsymbol{d}$: nematic vector



$$M_{\text{polar}} = \frac{U(1) \times S^2}{\mathbb{Z}_2}$$

invariant under

$$\begin{cases} \theta \to \theta + \pi \\ \boldsymbol{d} \to -\boldsymbol{d} \end{cases}$$

Fractional Vortex

$\pi_1(M) = \mathbb{Z}$ classifies vortices in the polar phase

$S^1 \to M$

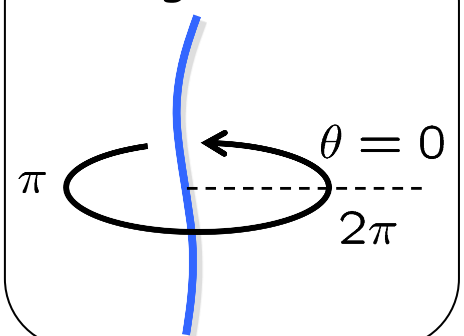
Quantum number of a vortex
= the circulation around the vortex in a unit of $\cancel{h/M}$ $h/2M$

$$\kappa = \oint_C \boldsymbol{v} \cdot d\boldsymbol{l} = \frac{\hbar}{M} \oint_C \nabla\theta \cdot d\boldsymbol{l} = \frac{\hbar}{M} \underbrace{\oint_C d\theta}_{\cancel{2\pi}\ \pi \times N}$$

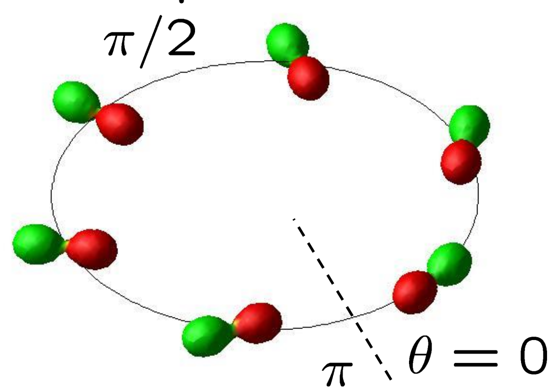
$$\boldsymbol{v} = \frac{\hbar}{M} \nabla\theta$$

$\because$ Z_2 symmetry

integer vortex



Half-quantum vortex

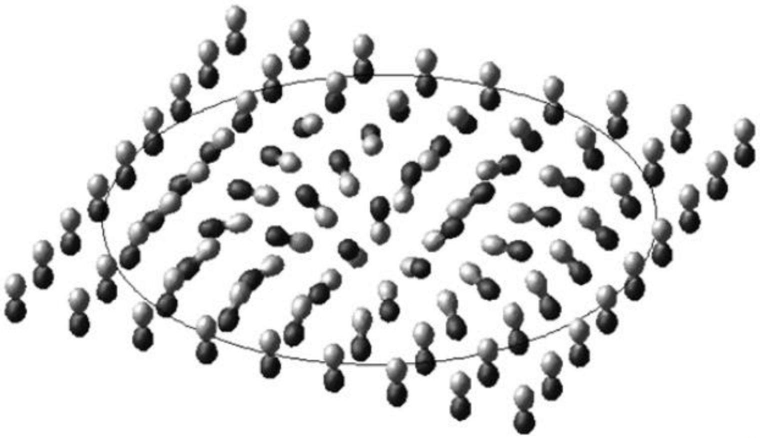


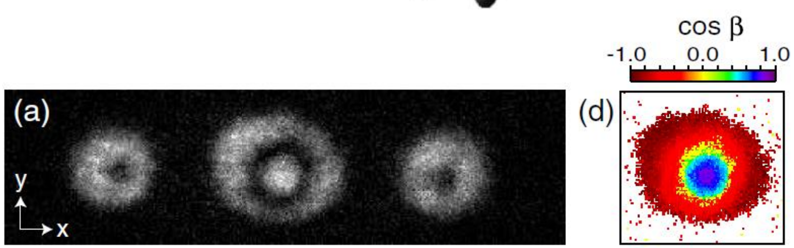
Excitations in Higher Dimensions

Skyrmion

$\pi_2(M) = \mathbb{Z}$

$$\boldsymbol{d}(\boldsymbol{r}) = \exp[-i\alpha(r)\hat{\boldsymbol{r}}\cdot\boldsymbol{F}]\hat{z}$$
$$\alpha(0) = \pi$$



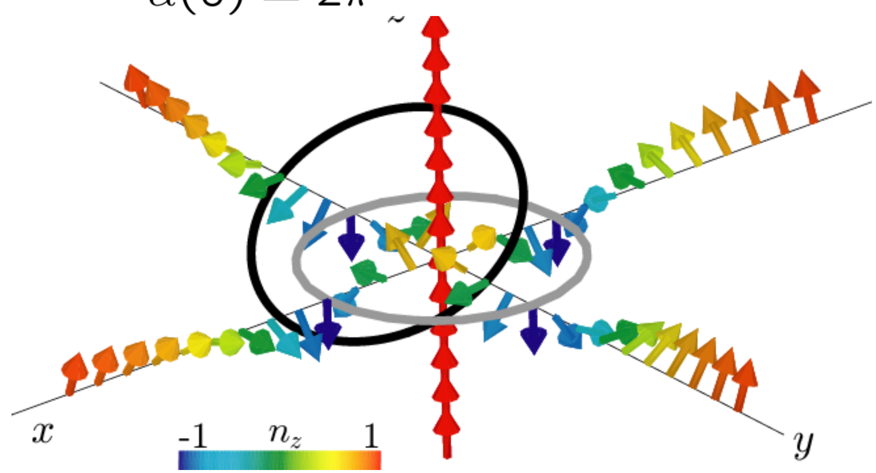


Choi, et. al. (PRL 2012)

Knot

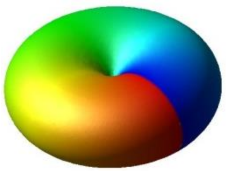
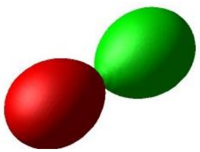
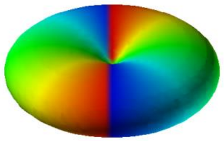
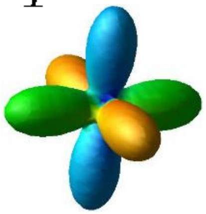
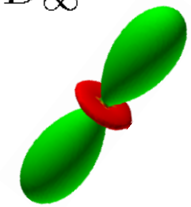
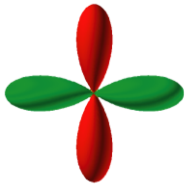
$\pi_3(M) = \mathbb{Z}$

$$\boldsymbol{d}(\boldsymbol{r}) = \exp[-i\alpha(r)\hat{\boldsymbol{r}}\cdot\boldsymbol{F}]\hat{z}$$
$$\alpha(0) = 2\pi$$



YK et. al. (PRL 2008)

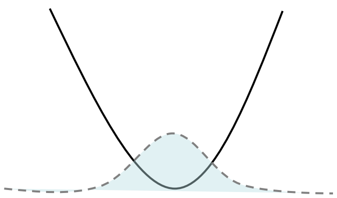
Symmetry and Topological Excitations

	spin-1		spin-2			
phase	ferro	polar	ferro	cyclic	uniaxial nematic	biaxial nematic
Symmetry	$U(1)$ 	D_∞ 	$U(1)$ 	T 	D_∞ 	D_4 
vortex $\pi_1(G/H)$	Z_2	1/2 vortex Z	Z_4	non-Abelian vortex $T^* \times Z$	1/2 vortex Z	non-Abelian vortex $D_4^* \times Z$
monopole/ 2D Skyrmion $\pi_2(G/H)$	0	2D Skyrmion Z	0	0	2D Skyrmion Z	0
Skyrmion $\pi_3(G/H)$	Skyrmion Z	knot Z	Skyrmion Z	Skyrmion Z	knot Z	Skyrmion Z

Spin-Singlet Pair Condensation

- strong confinement

  $\longrightarrow$ motional degrees of freedom is frozen



- three-mode system $\hat{a}_1,\ \hat{a}_0,\ \hat{a}_{-1}$

$$\hat{H} = \frac{\tilde{c}_0 + \tilde{c}_1}{2} \sum_{m_1 m_2} \hat{a}^\dagger_{m_1} \hat{a}^\dagger_{m_2} \hat{a}_{m_2} \hat{a}_{m_1} + \frac{\tilde{c}_1}{2} (2\hat{a}^\dagger_1 \hat{a}^\dagger_{-1} - \hat{a}^\dagger_0 \hat{a}^\dagger_0)(2\hat{a}_1 \hat{a}_{-1} - \hat{a}_0 \hat{a}_0)$$

- Starting from the m=0 state $\frac{1}{\sqrt{N!}}(\hat{a}^\dagger_0)^N |\mathrm{vac}\rangle$

  the number of m=1,-1 atoms increases via the term $\hat{a}^\dagger_1 \hat{a}^\dagger_{-1} \hat{a}_0 \hat{a}_0$

- Exact ground state: condensation of spin-singlet pairs

$$(2\hat{a}^\dagger_1 \hat{a}^\dagger_{-1} - \hat{a}^\dagger_0)^{N/2} |\mathrm{vac}\rangle$$

preserves SO(3) spin rotational symmetry

$$M = G/H = U(1)$$

Law et. al. (PRL 1998), Pu et al. (PRA 1999), Koashi & Ueda (PRL 2000)

Question

Is the spontaneous symmetry breaking really true
for a usual condensate in experiments ?

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- Bogoliubov analysis
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- Number conserving Bogoliubov analysis
  - Instability of zero energy mode

Bogoliubov Theory

- Gross-Pitaevskii equation

$$i\hbar\frac{\partial\psi}{\partial t}=\frac{\delta E}{\delta\psi^*}=\left[-\frac{\hbar^2}{2M}\nabla^2+V_{\mathrm{ext}}\right]\psi+c_0|\psi|^2\psi$$

- fluctuation around the stationary solution

$$\mu\varphi=\left[-\frac{\hbar^2}{2M}\nabla^2+V_{\mathrm{ext}}\right]\varphi+c_0|\varphi|^2\varphi$$

$$\psi(\boldsymbol{r})=(\varphi+\delta\psi)e^{-i\mu t},\quad \delta\psi(\boldsymbol{r})=\sum_\lambda[u_\lambda(\boldsymbol{r})e^{-i\omega_\lambda t}+v_\lambda^*(\boldsymbol{r})e^{i\omega_\lambda t}]$$

- eigenvalue equation (Bogoliubov equation)

$$\begin{pmatrix}\mathcal{L}+2c_0|\varphi|^2 & c_0\varphi^2\\ -c_0(\varphi^*)^2 & -\mathcal{L}-2c_0|\varphi|^2\end{pmatrix}\begin{pmatrix}u_\lambda(\boldsymbol{r})\\ v_\lambda(\boldsymbol{r})\end{pmatrix}=\hbar\omega_\lambda\begin{pmatrix}u_\lambda(\boldsymbol{r})\\ v_\lambda(\boldsymbol{r})\end{pmatrix}$$

$$\mathcal{L}=-\frac{\hbar^2}{2M}\nabla^2+V_{\mathrm{ext}}-\mu$$

non-Hermitian matrix

ω_λ can be a complex number

fluctuations grow exponentially

→Dynamical Instability

Microscopic Description

- atoms condense into the k=0 state in a uniform system

$$\hat{H} = \sum_{\boldsymbol{k}} \epsilon_{\boldsymbol{k}} \hat{a}^\dagger_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}} + \frac{c_0}{2\Omega} \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4} \hat{a}^\dagger_{\boldsymbol{k}_1} \hat{a}^\dagger_{\boldsymbol{k}_2} \hat{a}_{\boldsymbol{k}_3} \hat{a}_{\boldsymbol{k}_4} \delta_{\boldsymbol{k}_1+\boldsymbol{k}_2,\boldsymbol{k}_3+\boldsymbol{k}_4}$$

$$\hat{a}_{\boldsymbol{0}} \sim \sqrt{N}, \quad \hat{H} \sim \hat{H}_{\text{Bog}} = E_0 + \sum_{\boldsymbol{k}\neq\boldsymbol{0}} \left[(\epsilon_{\boldsymbol{k}} - \mu + 2c_0 n) \hat{a}^\dagger_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}} + \frac{c_0 n}{2} (\hat{a}_{\boldsymbol{k}} \hat{a}_{-\boldsymbol{k}} + \hat{a}^\dagger_{\boldsymbol{k}} \hat{a}^\dagger_{-\boldsymbol{k}}) \right]$$

- Bogoliubov transformation

$$\begin{pmatrix} \hat{a}_{\boldsymbol{k}} \\ \hat{a}^\dagger_{-\boldsymbol{k}} \end{pmatrix} = \begin{pmatrix} u_{\boldsymbol{k}} & v^*_{-\boldsymbol{k}} \\ v_{\boldsymbol{k}} & u^*_{-\boldsymbol{k}} \end{pmatrix} \begin{pmatrix} \hat{b}_{\boldsymbol{k}} \\ \hat{b}^\dagger_{-\boldsymbol{k}} \end{pmatrix}$$

$|u_{\boldsymbol{k}}|^2 - |v_{\boldsymbol{k}}|^2 = 1$ (canonical commutation relation for b-particles)

- H_{Bog} is diagonalized when u,v satisfy the Bogoliubov equation

$$\begin{pmatrix} \epsilon_{\boldsymbol{k}} + c_0 n & c_0 n \\ -c_0 n & -\epsilon_{\boldsymbol{k}} - c_0 n \end{pmatrix} \begin{pmatrix} u_{\boldsymbol{k}} \\ v_{\boldsymbol{k}} \end{pmatrix} = E_{\boldsymbol{k}} \begin{pmatrix} u_{\boldsymbol{k}} \\ v_{\boldsymbol{k}} \end{pmatrix} \Rightarrow \hat{H}_{\text{Bog}} = E_0 + \sum_{\boldsymbol{k}\neq\boldsymbol{0}} E_{\boldsymbol{k}} \hat{b}^\dagger_{\boldsymbol{k}} \hat{b}_{\boldsymbol{k}}$$

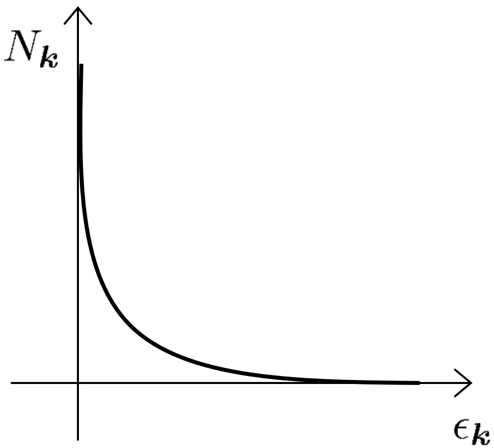
- Energy spectrum

$$E_{\boldsymbol{k}} = \sqrt{\epsilon_{\boldsymbol{k}}(\epsilon_{\boldsymbol{k}} + c_0 n)}, \quad u_{\boldsymbol{k}} = \sqrt{\frac{\epsilon_{\boldsymbol{k}} + c_0 n + E_{\boldsymbol{k}}}{2E_{\boldsymbol{k}}}}, \quad v_{\boldsymbol{k}} = -\sqrt{\frac{\epsilon_{\boldsymbol{k}} + c_0 n - E_{\boldsymbol{k}}}{2E_{\boldsymbol{k}}}}$$

Effects of Quantum Fluctuations

- Quantum depletion
  - Vacuum of Bogoliubov particles $\hat{b}_{\boldsymbol{k}}|\mathrm{vac_B}\rangle = 0$
  - number of non-condensed atoms

$$N_{\boldsymbol{k}} \equiv \langle \mathrm{vac_B}|\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}|\mathrm{vac_B}\rangle = |v_{\boldsymbol{k}}|^2 = \frac{\epsilon_{\boldsymbol{k}} + c_0 n - E_{\boldsymbol{k}}}{2E_{\boldsymbol{k}}}$$



  - total condensate depletion (3D)

$$\frac{N_{\mathrm{qd}}}{N} = \frac{\sum_{\boldsymbol{k}\neq\boldsymbol{0}} N_{\boldsymbol{k}}}{N} = \frac{8}{3}\sqrt{\frac{na^3}{\pi}}, \quad a = \frac{M}{4\pi\hbar^2}c_0$$

- Dynamical Instability
  - Quasi-particles for complex modes do not satisfy commutation relation

$$[\hat{b}_1, \hat{b}_1^\dagger] = [\hat{b}_2, \hat{b}_2^\dagger] = 0, \quad [\hat{b}_1, \hat{b}_2^\dagger] = 1$$

  - Using the "correct" quasi-particles,

$$\hat{H}_{\mathrm{Bog}} = E_0 + \sum E_{\boldsymbol{k}}\hat{b}^\dagger_{\boldsymbol{k}}\hat{b}_{\boldsymbol{k}} + (\mathrm{Im}E)(\hat{\beta}_1^\dagger\hat{\beta}_2^\dagger + \hat{\beta}_1\hat{\beta}_2)$$

Bogoliubov Theory for a Spinor BEC

- spin-1: internal degrees of freedom m=1,0,-1

$$\hat{H} = \sum_{\boldsymbol{k},m} \epsilon_{\boldsymbol{k}} \hat{a}^\dagger_{m\boldsymbol{k}} \hat{a}_{m\boldsymbol{k}} + \frac{1}{2\Omega} \sum_{\boldsymbol{k}_1,\cdots,\boldsymbol{k}_4} \sum_{m_1,\cdots,m_4} C^{m_1,m_2}_{m_4,m_3} \hat{a}^\dagger_{m_1\boldsymbol{k}_1} \hat{a}^\dagger_{m_2\boldsymbol{k}_2} \hat{a}_{m_3\boldsymbol{k}_3} \hat{a}_{m_4\boldsymbol{k}_4} \delta_{\boldsymbol{k}_1+\boldsymbol{k}_2,\boldsymbol{k}_3+\boldsymbol{k}_4}$$

$$C^{m_1,m_2}_{m_4,m_3} = c_0 \delta_{m_1,m_4} \delta_{m_2,m_3} + c_1 (\boldsymbol{F})_{m_1,m_4} (\boldsymbol{F})_{m_2,m_3}$$

invariant under
- U(1) gauge transformation
- SO(3) spin rotation

- Bogoliubov transformation

$$\hat{a}_{m\boldsymbol{k}} = \sum_n (U_{mn,\boldsymbol{k}} \hat{b}_{n\boldsymbol{k}} + V^*_{mn,-\boldsymbol{k}} \hat{b}^\dagger_{n,-\boldsymbol{k}})$$

$$\hat{a}^\dagger_{m,-\boldsymbol{k}} = \sum_n (V_{mn,\boldsymbol{k}'} \hat{b}_{n\boldsymbol{k}} + U^*_{mn,-\boldsymbol{k}} \hat{b}^\dagger_{n,-\boldsymbol{k}})$$

$$\boldsymbol{U}_{\boldsymbol{k}} = \begin{pmatrix} \boldsymbol{u}_{1\boldsymbol{k}} & \boldsymbol{u}_{0\boldsymbol{k}} & \boldsymbol{u}_{-1,\boldsymbol{k}} \end{pmatrix}$$

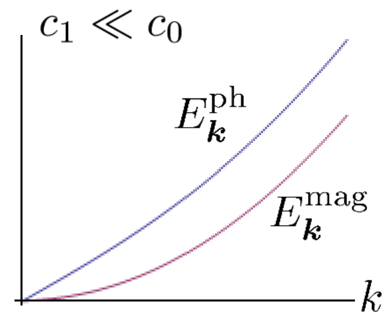
$$\boldsymbol{V}_{\boldsymbol{k}} = \begin{pmatrix} \boldsymbol{v}_{1\boldsymbol{k}} & \boldsymbol{v}_{0\boldsymbol{k}} & \boldsymbol{v}_{-1,\boldsymbol{k}} \end{pmatrix}$$

- Bogoliubov equation (6 x 6 matrix equation)

$$\begin{pmatrix} \boldsymbol{M}_{\boldsymbol{k}} & \boldsymbol{N}_{\boldsymbol{k}} \\ -\boldsymbol{N}^*_{-\boldsymbol{k}} & -\boldsymbol{M}^*_{-\boldsymbol{k}} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_{m\boldsymbol{k}} \\ \boldsymbol{v}_{m\boldsymbol{k}} \end{pmatrix} = E_{m\boldsymbol{k}} \begin{pmatrix} \boldsymbol{u}_{m\boldsymbol{k}} \\ \boldsymbol{v}_{m\boldsymbol{k}} \end{pmatrix}$$

Excitation Spectrum

- Condensate in the polar (m=0) state



- Energy spectrum

$$E_{\boldsymbol{k}}^{\mathrm{ph}} = \sqrt{\epsilon_{\boldsymbol{k}}(\epsilon_{\boldsymbol{k}} + c_0 n)}, \quad \hat{b}_{0\boldsymbol{k}} = u_{\boldsymbol{k}} \hat{a}_{0,\boldsymbol{k}} + v_{\boldsymbol{k}} \hat{a}^{\dagger}_{0,-\boldsymbol{k}}$$

$$E_{\boldsymbol{k}}^{\mathrm{mag}} = \sqrt{\epsilon_{\boldsymbol{k}}(\epsilon_{\boldsymbol{k}} + c_1 n)}, \quad \hat{b}_{\pm 1,\boldsymbol{k}} = u_{\boldsymbol{k}} \frac{\hat{a}_{1,\boldsymbol{k}} \pm \hat{a}_{-1,\boldsymbol{k}}}{\sqrt{2}} + v_{\boldsymbol{k}} \frac{\hat{a}^{\dagger}_{1,-\boldsymbol{k}} \pm \hat{a}^{\dagger}_{-1,-\boldsymbol{k}}}{\sqrt{2}}$$

- $\hat{H}_{\mathrm{Bog}} = E_0 + \sum_{(\boldsymbol{k},m)\neq(\boldsymbol{0},0)} E_{m\boldsymbol{k}} \hat{b}^{\dagger}_{m\boldsymbol{k}} \hat{b}_{m\boldsymbol{k}}$ $\longrightarrow$ condensate is stable?

- Quantum Depletion

$$N_{\boldsymbol{k}}^{\mathrm{ph}} = \frac{\epsilon_{\boldsymbol{k}} + c_0 n - E_{\boldsymbol{k}}^{\mathrm{ph}}}{2E_{\boldsymbol{k}}^{\mathrm{ph}}} \qquad N_{\boldsymbol{k}}^{\mathrm{mag}} = \frac{\epsilon_{\boldsymbol{k}} + c_1 n - E_{\boldsymbol{k}}^{\mathrm{mag}}}{2E_{\boldsymbol{k}}^{\mathrm{mag}}}$$

diverges at k=0

Macroscopic occupation in the m=$\pm$1 & k=0 states?

What's wrong?

Condensate in the m=0 state is unstable.
Goldstone magnon seems to cause this instability.

What's wrong with the Bogoliubov theory?

It does not conserve the total number of atoms:

$$\hat{a}_{\mathbf{0}} \sim \sqrt{N}$$

Number-conserving Bogoliubov theory
can deal with this problem.

Number-conserving Bogoliubov Theory

Leggett, RMP (2001)

- Mean-field theory (Hartree):

  All atoms are condensed in the same single-particle state

$$\Psi_N(\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_N) = \prod_{i=1}^{N} \varphi_1(\boldsymbol{r}_i) \qquad |\Psi\rangle = \frac{1}{\sqrt{N!}} (\hat{a}^\dagger_{0\boldsymbol{0}})^N |\mathrm{vac}\rangle$$

- All pairs of atoms are condensed in the same two-particle state

  Two-body correlation is taken into account.

$$\Psi_N(\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_N) = \mathcal{S} \prod_{i<j}^{N} \varphi_2(\boldsymbol{r}_i, \boldsymbol{r}_j)$$

- Ansatz

$$|\Psi(t)\rangle = \frac{\mathcal{N}(t)}{\sqrt{N!}} \left[(\hat{a}^\dagger_{0\boldsymbol{0}})^2 - \sum_{\boldsymbol{k}\neq\boldsymbol{0}} \Lambda_{\boldsymbol{k}}(t) \hat{a}^\dagger_{0\boldsymbol{k}} \hat{a}^\dagger_{0,-\boldsymbol{k}} - \sum_{\boldsymbol{k}} \Lambda'_{\boldsymbol{k}}(t) \hat{a}^\dagger_{1\boldsymbol{k}} \hat{a}^\dagger_{-1,-\boldsymbol{k}} \right]^{N/2} |\mathrm{vac}\rangle$$

$$= \frac{\mathcal{N}(t)}{\sqrt{N!}} \left[(\hat{\alpha}^\dagger_{0\boldsymbol{0}})^2 - \sum_{m=\pm1} \lambda_{m\boldsymbol{0}}(t) (\hat{\alpha}^\dagger_{m\boldsymbol{0}})^2 - \sum_{m,\boldsymbol{k}\neq 0} \lambda_{m\boldsymbol{k}}(t) \hat{\alpha}^\dagger_{m\boldsymbol{k}} \hat{\alpha}^\dagger_{m,-\boldsymbol{k}} \right]^{N/2} |\mathrm{vac}\rangle$$

$$\hat{\alpha}_{\pm1,\boldsymbol{k}} \equiv \frac{\hat{a}_{1\boldsymbol{k}} \pm \hat{a}_{-1\boldsymbol{k}}}{\sqrt{2}}, \quad \hat{\alpha}_{0\boldsymbol{k}} \equiv \hat{a}_{0\boldsymbol{k}} \qquad |\lambda_{m\boldsymbol{k}}| < 1 \ (N_{m\boldsymbol{k}} \ll N)$$

Equation of Motion

- Functional action

$$S[\{\lambda_{m\boldsymbol{k}}, \lambda^*_{m\boldsymbol{k}}\}] = \int dt \, \langle \Psi(t)| \, \hat{H} - i\hbar \frac{d}{dt} \, |\Psi(t)\rangle$$

- Equation of motion for $\lambda_{m\boldsymbol{k}}$

$$\frac{\delta S[\{\lambda_{m\boldsymbol{k}}, \lambda^*_{m\boldsymbol{k}}\}]}{\delta \lambda^*_{m\boldsymbol{k}}} = 0 \quad \Rightarrow \quad \boxed{\begin{gathered} -i\hbar \dot{\lambda}_{m\boldsymbol{k}} = B_m(\lambda^2_{m\boldsymbol{k}} + 1) - 2(\epsilon_{\boldsymbol{k}} + A_m)\lambda_{m\boldsymbol{k}} \\ A_{\pm 1} = c_1 n, \ A_0 = c_0 n, \ B_{\pm 1} = \pm c_1 n, \ B_0 = c_0 n \end{gathered}}$$

$$\mathcal{N}^2 = \prod_{m'=\pm 1} \sqrt{1 - |\lambda_{m'\boldsymbol{0}}|^2} \prod_{m,\boldsymbol{k}>0} (1 - |\lambda_{m\boldsymbol{k}}|^2) + O(1/N)$$

$$N_{m\boldsymbol{k}} = \langle \hat{\alpha}^\dagger_{m\boldsymbol{k}} \hat{\alpha}_{m\boldsymbol{k}} \rangle = \frac{|\lambda_{m\boldsymbol{k}}|^2}{1 - |\lambda_{m\boldsymbol{k}}|^2} + O(1/N)$$

$$\langle \hat{\alpha}^\dagger_{0\boldsymbol{0}} \hat{\alpha}_{0\boldsymbol{0}} \hat{\alpha}^\dagger_{m\boldsymbol{k}} \hat{\alpha}_{m\boldsymbol{k}} \rangle = N \left[\frac{|\lambda_{m\boldsymbol{k}}|^2}{1 - |\lambda_{m\boldsymbol{k}}|^2} + O(1/N) \right]$$

$$\langle \hat{\alpha}^\dagger_{m\boldsymbol{k}} \hat{\alpha}^\dagger_{m,-\boldsymbol{k}} \hat{\alpha}_{0\boldsymbol{0}} \hat{\alpha}_{0\boldsymbol{0}} \rangle = -N \left[\frac{\lambda^*_{m\boldsymbol{k}}}{1 - |\lambda_{m\boldsymbol{k}}|^2} + O(1/N) \right]$$

Dynamics: Quantum Fluctuation

- Starting from $\lambda_{m\boldsymbol{k}} = 0$

$$\lambda_{m\boldsymbol{k}}(t) = \frac{B_m \sin(E_{m\boldsymbol{k}}t/\hbar)}{(\epsilon_{\boldsymbol{k}} + A_{m\boldsymbol{k}})\sin(E_{m\boldsymbol{k}}t/\hbar) - iE_{m\boldsymbol{k}}\cos(E_{m\boldsymbol{k}}t/\hbar)}$$

$$E_{m\boldsymbol{k}} = \sqrt{(\epsilon_{\boldsymbol{k}} + A_m + B_m)(\epsilon_{\boldsymbol{k}} + A_m - B_m)}$$

$$A_0 = c_0 n, \ B_0 = c_0 n$$
$$A_{\pm 1} = c_1 n, \ B_{\pm 1} = \pm c_1 n$$

$\Rightarrow$

$$E_{0\boldsymbol{k}} = E^{\mathrm{ph}}_{\boldsymbol{k}} = \sqrt{\epsilon_{\boldsymbol{k}}(\epsilon_{\boldsymbol{k}} + c_0 n)}$$
$$E_{\pm 1,\boldsymbol{k}} = E^{\mathrm{mag}}_{\boldsymbol{k}} = \sqrt{\epsilon_{\boldsymbol{k}}(\epsilon_{\boldsymbol{k}} + c_1 n)}$$

$$N_{m\boldsymbol{k}}(t) \equiv \frac{|\lambda_{m\boldsymbol{k}}|^2}{1 - |\lambda_{m\boldsymbol{k}}|^2} = \left| \frac{B_m}{E_{m\boldsymbol{k}}} \sin\left(\frac{E_{m\boldsymbol{k}}t}{\hbar}\right)\right|^2$$

$$E_{\pm 1,\mathbf{0}} = 0$$

$$N_{\pm 1,\mathbf{0}}(t) = \left(\frac{t}{\tau_0}\right)^2 \quad \tau_0 = \frac{\hbar}{B_m} = \frac{\hbar}{c_1 n}$$

algebraic growth

$E_{m\boldsymbol{k}}$: complex

$$N_{m\boldsymbol{k}}(t) = \frac{B_m}{|E_{m\boldsymbol{k}}|}\sinh^2\left(\frac{E_{m\boldsymbol{k}}t}{\hbar}\right)$$

exponential growth
(Dynamical Instability)

Quantum Depletion

- Stationary solution for the equation of motion

$$\lambda_{m\boldsymbol{k}} = \frac{\epsilon_{\boldsymbol{k}} + A_m - E_{m\boldsymbol{k}}}{B_m}$$

$$N_{m\boldsymbol{k}} = \frac{|\lambda_{m\boldsymbol{k}}|^2}{1 - |\lambda_{m\boldsymbol{k}}|^2} = \frac{\epsilon_{\boldsymbol{k}} + A_m - E_{m\boldsymbol{k}}}{2E_{m\boldsymbol{k}}}$$

- reproduces the quantum depletion obtained in the conventional Bogoliubov theory

$$N_{\boldsymbol{k}}^{\mathrm{ph}} = \frac{\epsilon_{\boldsymbol{k}} + c_0 n - E_{\boldsymbol{k}}^{\mathrm{ph}}}{2E_{\boldsymbol{k}}^{\mathrm{ph}}}$$

$$N_{\boldsymbol{k}}^{\mathrm{mag}} = \frac{\epsilon_{\boldsymbol{k}} + c_1 n - E_{\boldsymbol{k}}^{\mathrm{mag}}}{2E_{\boldsymbol{k}}^{\mathrm{mag}}}$$

- Fraction in the m=$\pm$1 & k=0 states diverges

$\longrightarrow$ macroscopic number of atoms exist in these states

Dynamics: Thermal Fluctuation

- start from

$$\lambda_{m\boldsymbol{k}}(0) = e^{i\theta}\sqrt{\frac{N_{\rm ex}}{N_{\rm ex}+1}} \qquad N_{m\boldsymbol{k}}(0) = N_{\rm ex}$$

- take an average over θ (thermal fluctuation)

$$\bar{N}_{m\boldsymbol{k}}(t) = \frac{1}{2\pi}\int_0^{2\pi} d\theta N_{m\boldsymbol{k}}(t) = N_{\rm ex} + (2N_{\rm ex}+1)\left|\frac{B_m}{E_{m\boldsymbol{k}}}\sin\left(\frac{E_{m\boldsymbol{k}}t}{\hbar}\right)\right|^2$$

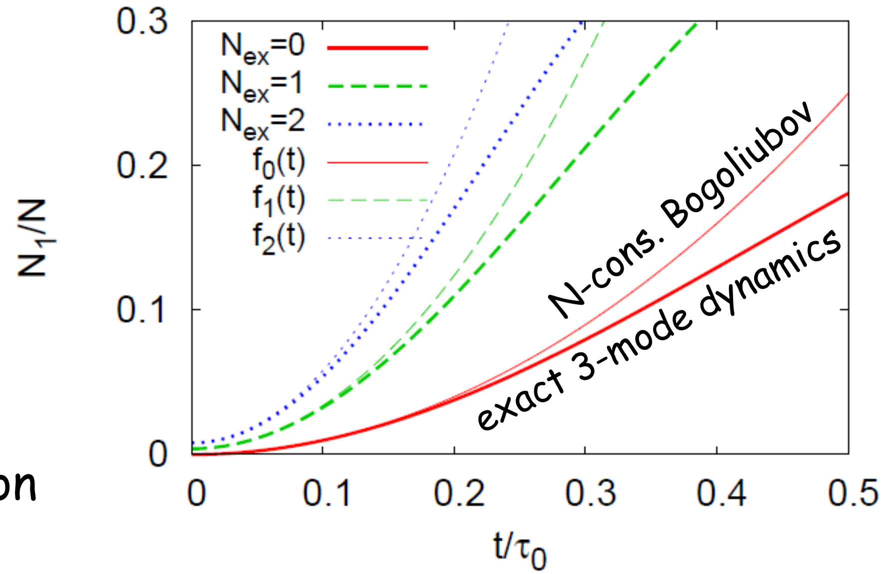
- zero-energy mode

$$\bar{N}_{m\boldsymbol{k}}(t) = N_{\rm ex} + \left(\frac{t}{\tau_0/\sqrt{2N_{\rm ex}+1}}\right)^2$$

grow faster
(Bosonic stimulation)

↓

Growth of the Goldstone magnon
can be observed.



Possible Experimental Scheme

- $N=10^6$ of Na atoms in a trap with frequency 300 Hz ➡ $\tau_0 = 1.7$ s
- If N_{ex}=5, $\tau_0 \Rightarrow \tau_0/\sqrt{2N_{\rm ex}+1} = 0.52$ s short enough to be observed

Possible experimental scheme

- Prepare m=0 condensate at q>>0

$$E_{0\boldsymbol{k}} = E_{\boldsymbol{k}}^{\rm ph} = \sqrt{\epsilon_{\boldsymbol{k}}(\epsilon_{\boldsymbol{k}} + c_0 n)}$$

$$E_{\pm1,\boldsymbol{k}} = E_{\boldsymbol{k}}^{\rm mag} = \sqrt{(\epsilon_{\boldsymbol{k}} + q)(\epsilon_{\boldsymbol{k}} + q + c_1 n)}$$

quadratic Zeeman effect

$$\hat{H}_Z = q\sum_{m\boldsymbol{k}} m^2 \hat{a}^\dagger_{m\boldsymbol{k}}\hat{a}_{m\boldsymbol{k}}$$

- Thermally excited atoms in the m=$\pm$1, k=0 states

$$N_{\pm1,\boldsymbol{0}} = \frac{1}{\exp(\sqrt{q(q+c_1 n)}/k_{\rm B}T)+1}$$

N_{ex} is tunable by changing the initial q $\propto$ B^2

- Quench q$\to$0
- The growth speed of the Goldstone magnon depends on the initial q

Conclusion & Future Prospect

- Goldstone magnon in a spinor BEC causes instability so as to recover the symmetry of the Hamiltonian.
- This is a general feature of Goldstone modes in a multi-component system as long as the conservation low prohibits the Goldstone mode to grow.
- Thermal fluctuation accelerates the growth dynamics due to bosonic stimulation.

- Does the spin-singlet-pair condensation occur in a large system?
- Stability of the pair condensation against nonzero k fluctuations should be addressed.

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