Pairing states in Kane-Mele-based systems

Jun Goryo (Hirosaki Univ.)

Spin-orbit coupling and pairing symmetry

Cf) chiral p-wave superconductor Sr_2RuO_4

possible spin triplet states in D_{4h}

Sigrist & Ueda RMP ('91)

Γ	$\boldsymbol{d}(\boldsymbol{k})$
A_{1u}	$\hat{x}\hat{k}_x + \hat{y}\hat{k}_y$
A_{2u}	$\hat{x}\hat{k}_y - \hat{y}\hat{k}_x$
B_{1u}	$\hat{x}\hat{k}_x - \hat{y}\hat{k}_y$
B_{1u}	$\hat{x}\hat{k}_y + \hat{y}\hat{k}_x$
E_u (chiral states)	$\hat{z}(\hat{k}_x \pm i\hat{k}_y)$

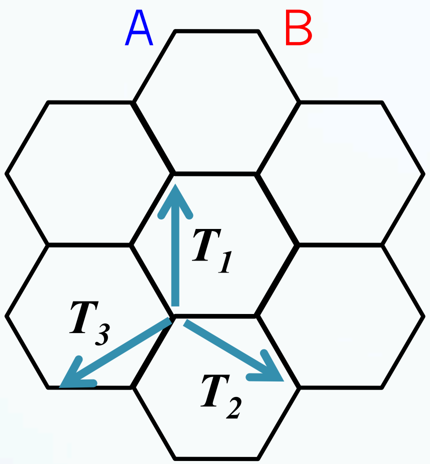
- These five states have degenerated condensation energy
- The degeneracy is lifted by SOC

⇨ **Chiral p-wave**

Ng&Sigrist ('99)
Yanase&Ogata('03)

Spin-orbit coupling and band topology

Kane-Mele insulator C.L. Kane and E. Mele PRL('05)('05)

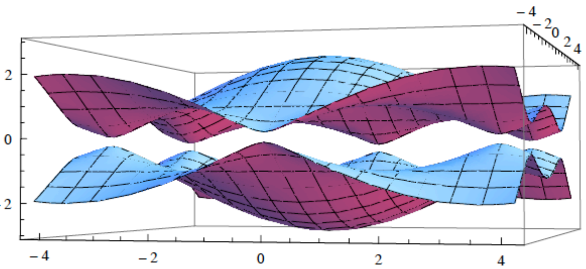


H= NN hopping + **SOC**

$$\hat{\mathcal{H}}^{\mathrm{SOC}}_{\boldsymbol{k}} = \lambda_{SO} \sum_{i=1,2,3} \sin \boldsymbol{k} \cdot \boldsymbol{T}_i \underline{\hat{\sigma}_z \otimes \hat{s}_z}$$

sublattice (A,B) spin (↑, ↓)

band gap $\propto \lambda_{SO}$



$\lambda_{SO}/t = 0.01$

non-trivial topology

Spin Chern #

$$= \int_{BZ} \frac{d^2k}{(2\pi)^2 i} \mathrm{rot} \left\{ \langle u_{\boldsymbol{k}\uparrow} | \nabla_{\boldsymbol{k}} | u_{\boldsymbol{k}\uparrow} \rangle - \langle u_{\boldsymbol{k}\downarrow} | \nabla_{\boldsymbol{k}} | u_{\boldsymbol{k}\downarrow} \rangle \right\}$$

$$= \frac{\lambda_{\mathrm{SO}}}{|\lambda_{\mathrm{SO}}|}$$

→ The helical edge state

QSHE

Our issue:

How does the topological SOC of KM model affect **the pairing states** ?

1. Cooperon in KM insulator

with S. Tsuchiya, E. Arahata, and M. Sigrist

Cf) Nozieres-Pistoresi ('98)

Band insulator + attractive U
Increasing U ⇒ Cooperon excitations
Increasing more... ⇒ SC due to Cooperon condensation

The KM SOC favors the helical p-wave Cooperon channel

(2. SrPtAs: an example of KM metal and its superconductivity)

with M. Fischer, A. Schnyder, T. Neupert, R. Thomale...
Y. Imai, and M. Sigrist

Cooperons in Kane-Mele model

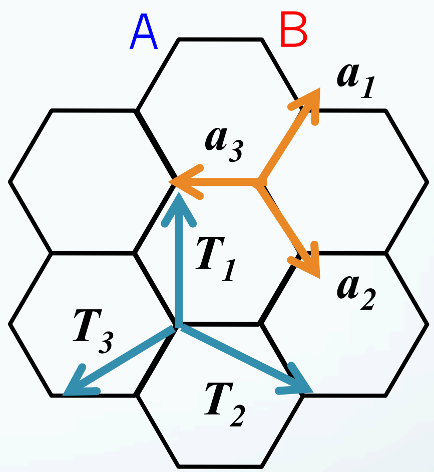
With S. Tsuchiya, E. Arahata, and M. Sigrist

$$H_{KM} = \sum_{\boldsymbol{k}} \left\{ \Psi^\dagger_{\boldsymbol{k}\uparrow} \begin{pmatrix} \lambda_{\boldsymbol{k}} & t_{\boldsymbol{k}} \\ t^*_{\boldsymbol{k}} & -\lambda_{\boldsymbol{k}} \end{pmatrix} \Psi_{\boldsymbol{k}\uparrow} + \Psi^\dagger_{\boldsymbol{k}\downarrow} \begin{pmatrix} -\lambda_{\boldsymbol{k}} & t_{\boldsymbol{k}} \\ t^*_{\boldsymbol{k}} & \lambda_{\boldsymbol{k}} \end{pmatrix} \Psi_{\boldsymbol{k}\downarrow} \right\}$$

Fermion OP $\Psi_{\boldsymbol{k}s} = \begin{pmatrix} A_{\boldsymbol{k}s} \\ B_{\boldsymbol{k}s} \end{pmatrix}$

NN hopping $t_{\boldsymbol{k}} = t \sum_i e^{i\boldsymbol{k}\cdot\boldsymbol{a}_i}$

SOC $\lambda_{\boldsymbol{k}} = \lambda_{SO} \sum_i \sin \boldsymbol{k} \cdot \boldsymbol{T}_i$



When we introduce...

$$H_{\text{on}} = -U \sum_i n_{i\uparrow} n_{i\downarrow}$$

U>Uc → S-wave condensate

Cooperons in Kane-Mele model

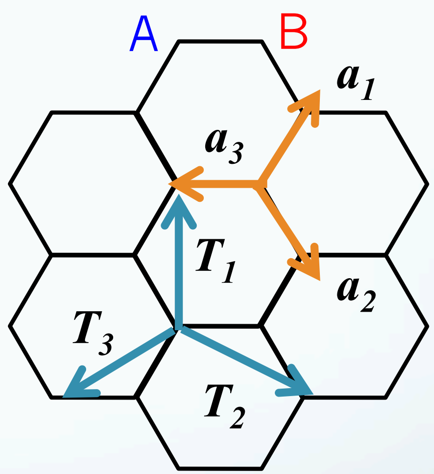
With S. Tsuchiya, E. Arahata, and M. Sigrist

$$H_{KM} = \sum_{\boldsymbol{k}} \left\{ \Psi^\dagger_{\boldsymbol{k}\uparrow} \begin{pmatrix} \lambda_{\boldsymbol{k}} & t_{\boldsymbol{k}} \\ t^*_{\boldsymbol{k}} & -\lambda_{\boldsymbol{k}} \end{pmatrix} \Psi_{\boldsymbol{k}\uparrow} + \Psi^\dagger_{\boldsymbol{k}\downarrow} \begin{pmatrix} -\lambda_{\boldsymbol{k}} & t_{\boldsymbol{k}} \\ t^*_{\boldsymbol{k}} & \lambda_{\boldsymbol{k}} \end{pmatrix} \Psi_{\boldsymbol{k}\downarrow} \right\}$$

Fermion OP $$\Psi_{\boldsymbol{k}s} = \begin{pmatrix} A_{\boldsymbol{k}s} \\ B_{\boldsymbol{k}s} \end{pmatrix}$$

NN hopping $$t_{\boldsymbol{k}} = t \sum_i e^{i\boldsymbol{k}\cdot\boldsymbol{a}_i}$$

SOC $$\lambda_{\boldsymbol{k}} = \lambda_{SO} \sum_i \sin \boldsymbol{k}\cdot\boldsymbol{T}_i$$



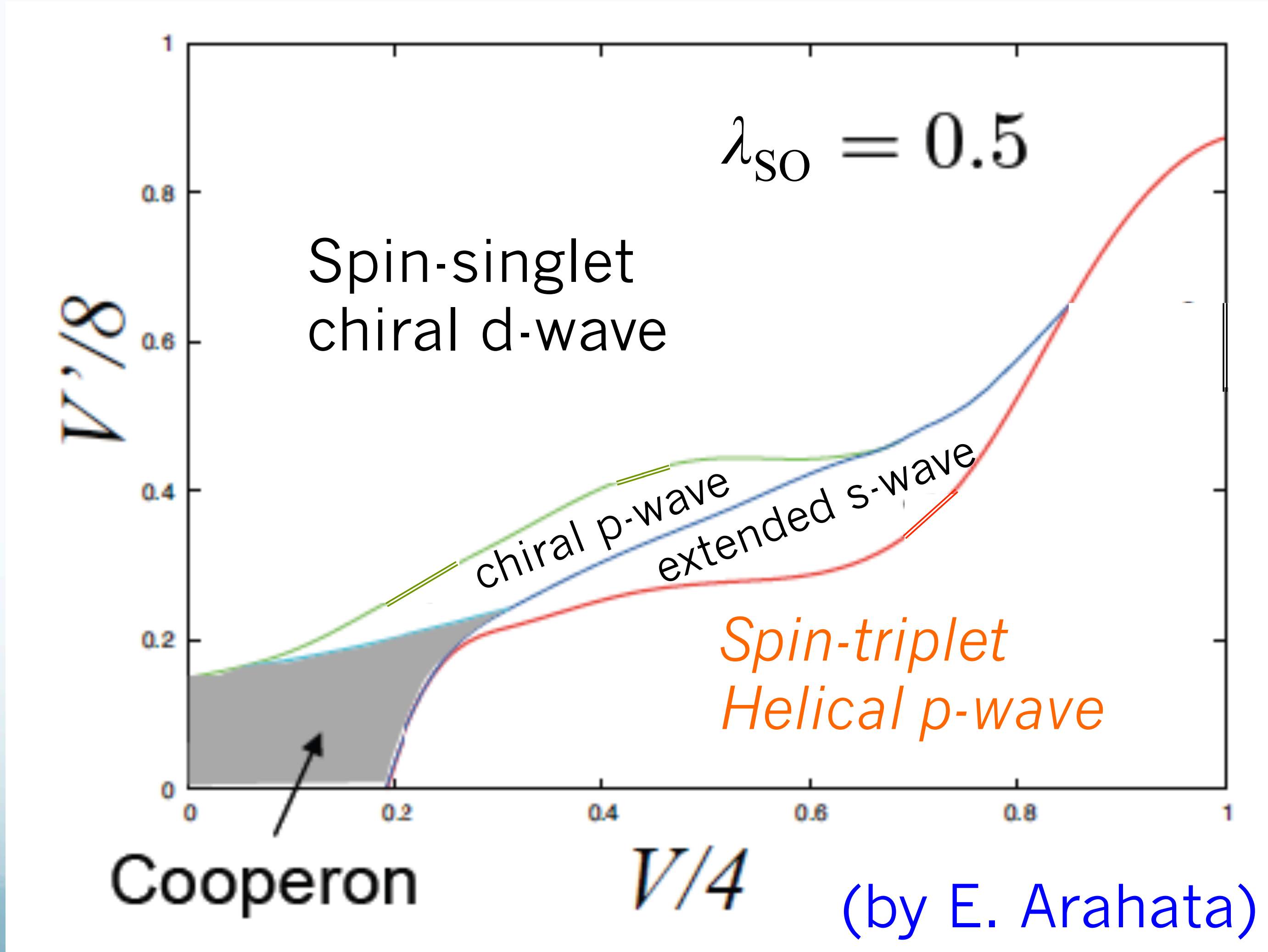
When we introduce...

$$H_{\text{int}} = -V \sum_{\langle i,j \rangle} n_i n_j - V' \sum_{\langle\langle i,j \rangle\rangle} n_i n_j$$

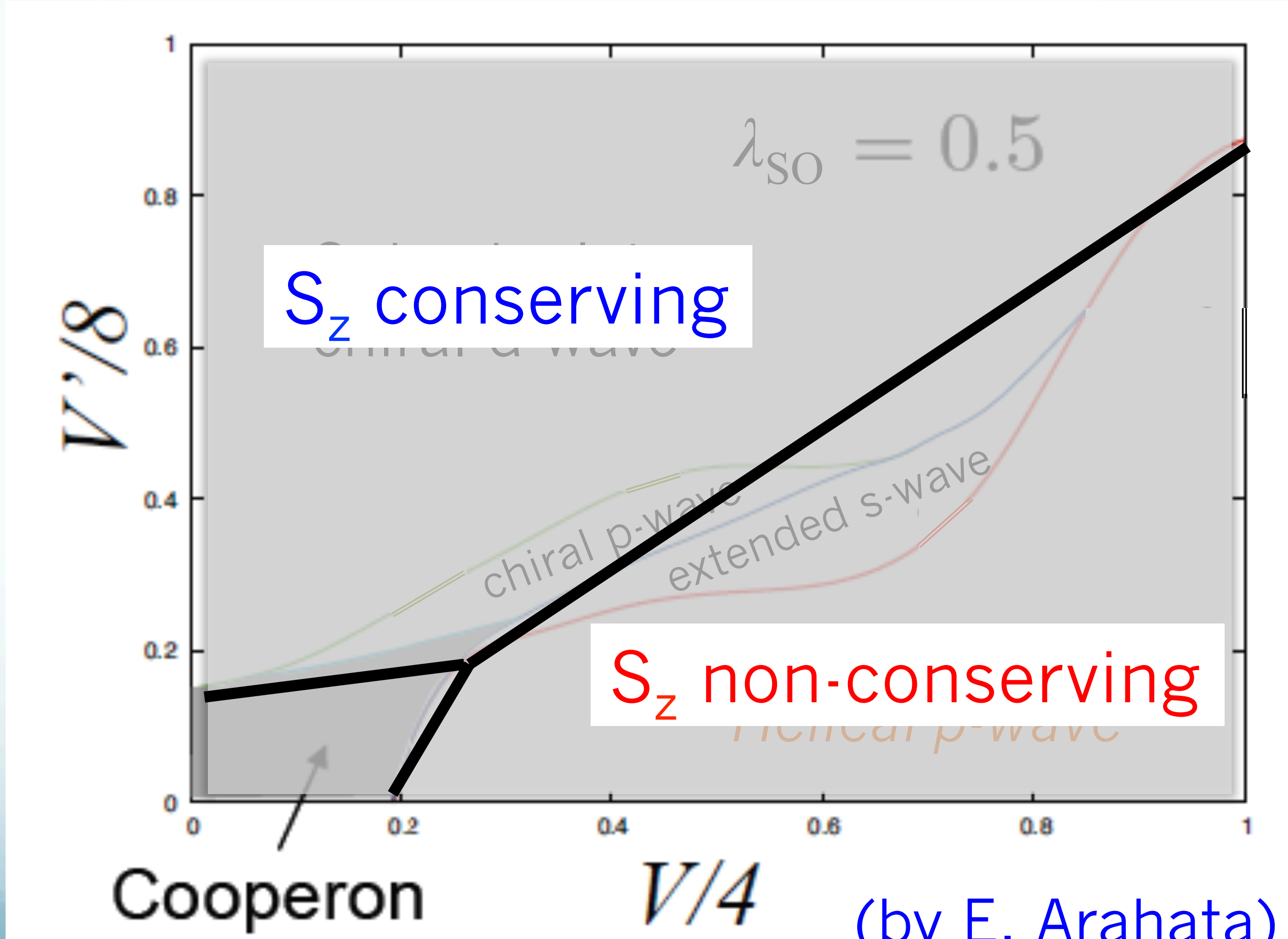
V>Vc
V>Vc'

? ? ?

Analysis of the Gap equation at T=0 with V (*NN*) and V' (*NNN* attraction)



Analysis of the Gap equation at T=0 with V (*NN*) and V' (*NNN* attraction)



(by E. Arahata)

Eigenstates of H_{KM}

$$\epsilon_{\boldsymbol{k}}^{(+)} = +\sqrt{|t_{\boldsymbol{k}}|^2 + \lambda_{\boldsymbol{k}}^2} \equiv \epsilon_{\boldsymbol{k}}$$

$$\epsilon_{\boldsymbol{k}}^{(-)} = -\sqrt{|t_{\boldsymbol{k}}|^2 + \lambda_{\boldsymbol{k}}^2} = -\epsilon_{\boldsymbol{k}}$$

$$|\boldsymbol{k} \uparrow +\rangle = \begin{pmatrix} u_{\boldsymbol{k}} e^{i\phi_{\boldsymbol{k}}} \\ v_{\boldsymbol{k}} \end{pmatrix}$$

$$|\boldsymbol{k} \downarrow +\rangle = \begin{pmatrix} v_{\boldsymbol{k}} \\ u_{\boldsymbol{k}} e^{-i\phi_{\boldsymbol{k}}} \end{pmatrix}$$

$$|\boldsymbol{k} \uparrow -\rangle = \begin{pmatrix} v_{\boldsymbol{k}} \\ -u_{\boldsymbol{k}} e^{-i\phi_{\boldsymbol{k}}} \end{pmatrix}$$

$$|\boldsymbol{k} \downarrow -\rangle = \begin{pmatrix} u_{\boldsymbol{k}} e^{i\phi_{\boldsymbol{k}}} \\ -v_{\boldsymbol{k}} \end{pmatrix}$$

2-fold (spin) degeneracy

$$u_{\boldsymbol{k}} = \sqrt{\frac{1}{2}\left(1 + \frac{\lambda_{\boldsymbol{k}}}{\epsilon_{\boldsymbol{k}}}\right)}$$

$$v_{\boldsymbol{k}} = \sqrt{\frac{1}{2}\left(1 - \frac{\lambda_{\boldsymbol{k}}}{\epsilon_{\boldsymbol{k}}}\right)}$$

$$\phi_{\boldsymbol{k}} = \arg(t_{\boldsymbol{k}})$$

- Pairing between two states around K and K’ points would be dominant due to the narrow gap

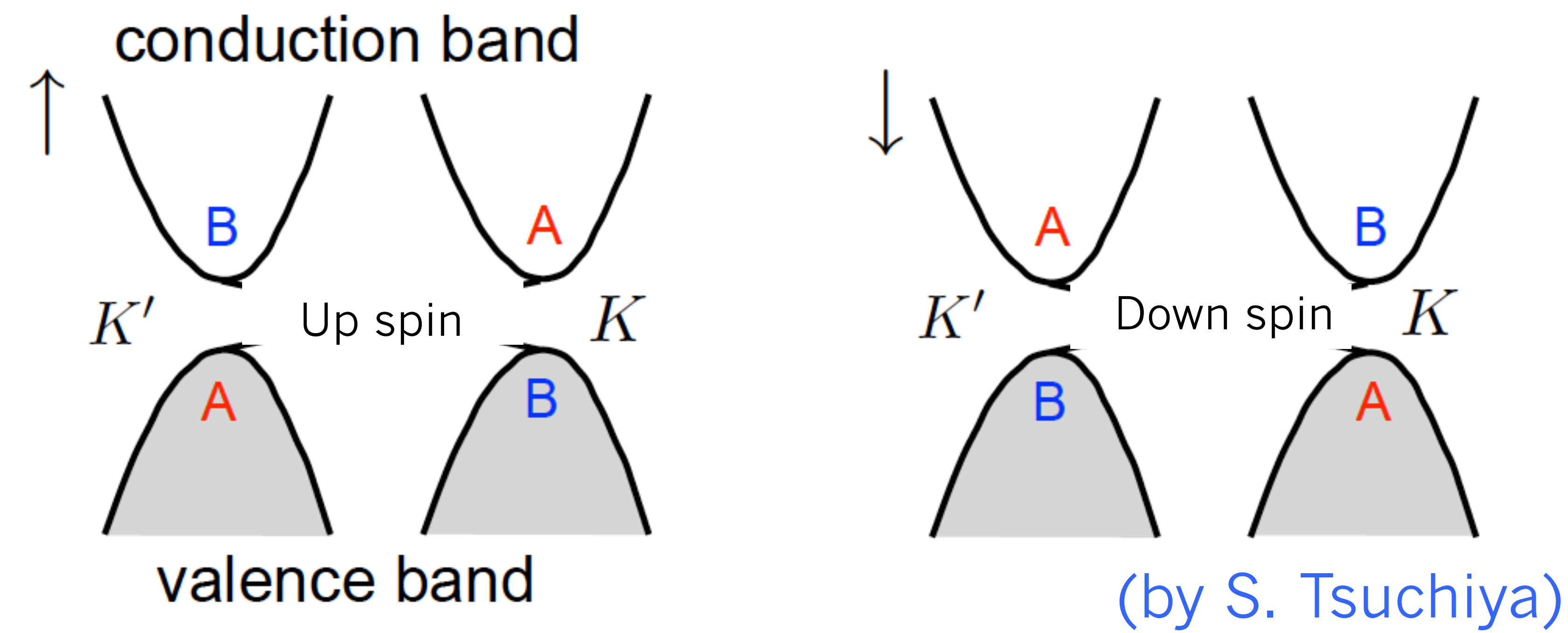
↓

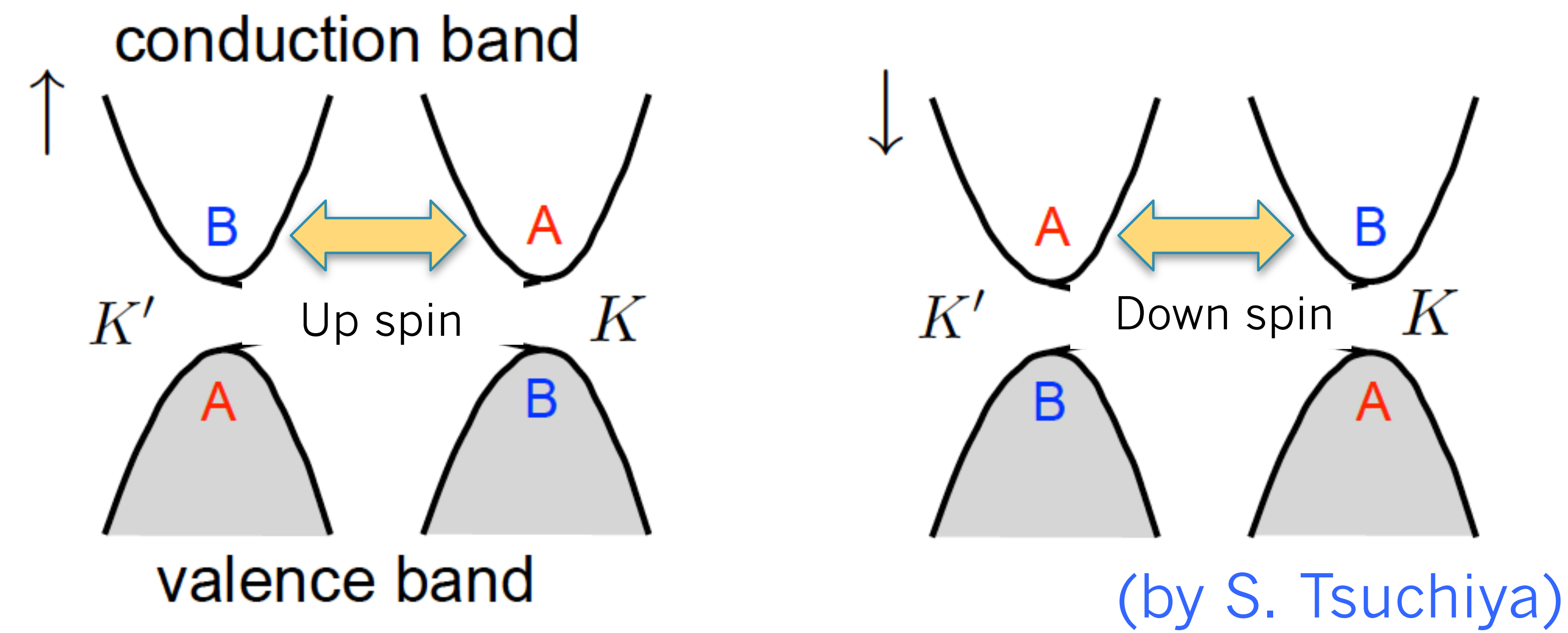
focus on the behavior around K and K’ points

	around K	around K'
$\lvert t_{\boldsymbol{k}} \rvert$	0	0
$\phi_{\boldsymbol{k}} = \arg(t_{\boldsymbol{k}})$	$\tan^{-1} \frac{\delta k_y}{\delta k_x} \equiv \theta_{\delta \boldsymbol{k}}$	$-\tan^{-1} \frac{\delta k_y}{\delta k_x} = -\theta_{\delta \boldsymbol{k}}$
$\lambda_{\boldsymbol{k}}$	λ_{SO}	$-\lambda_{\mathrm{SO}}$
$\epsilon_{\boldsymbol{k}}$	$\lvert \lambda_{\mathrm{SO}} \rvert$	$\lvert \lambda_{\mathrm{SO}} \rvert$
	$\cdot\ \lambda_{\mathrm{SO}} > 0$	$\cdot\ \lambda_{\mathrm{SO}} > 0$
$u_{\boldsymbol{k}} = \sqrt{\frac{1}{2}\left(1 + \frac{\lambda_{\boldsymbol{k}}}{\epsilon_{\boldsymbol{k}}}\right)}$	1	0
$v_{\boldsymbol{k}} = \sqrt{\frac{1}{2}\left(1 - \frac{\lambda_{\boldsymbol{k}}}{\epsilon_{\boldsymbol{k}}}\right)}$	0	1

	around K	around K'
$\lvert t_{\boldsymbol{k}} \rvert$	0	0
$\phi_{\boldsymbol{k}} = \arg(t_{\boldsymbol{k}})$	$\tan^{-1} \frac{\delta k_y}{\delta k_x} \equiv \theta_{\delta \boldsymbol{k}}$	$-\tan^{-1} \frac{\delta k_y}{\delta k_x} = -\theta_{\delta \boldsymbol{k}}$
$\lambda_{\boldsymbol{k}}$	λ_{SO}	$-\lambda_{\mathrm{SO}}$
$\epsilon_{\boldsymbol{k}}$	$\lvert\lambda_{\mathrm{SO}}\rvert$	$\lvert\lambda_{\mathrm{SO}}\rvert$
	$\cdot\ \lambda_{\mathrm{SO}} > 0$	$\cdot\ \lambda_{\mathrm{SO}} > 0$
$u_{\boldsymbol{k}} = \sqrt{\frac{1}{2}\left(1 + \frac{\lambda_{\boldsymbol{k}}}{\epsilon_{\boldsymbol{k}}}\right)}$	1	0
$v_{\boldsymbol{k}} = \sqrt{\frac{1}{2}\left(1 - \frac{\lambda_{\boldsymbol{k}}}{\epsilon_{\boldsymbol{k}}}\right)}$	0	1
$\lvert \boldsymbol{k} \uparrow +\rangle = \begin{pmatrix} u_{\boldsymbol{k}} e^{i\phi_{\boldsymbol{k}}} \\ v_{\boldsymbol{k}} \end{pmatrix}$	$\begin{pmatrix} e^{-i\theta} \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$\lvert \boldsymbol{k} \downarrow +\rangle = \begin{pmatrix} v_{\boldsymbol{k}} \\ u_{\boldsymbol{k}} e^{-i\phi_{\boldsymbol{k}}} \end{pmatrix}$	$\begin{pmatrix} 0 \\ e^{i\theta} \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$\lvert \boldsymbol{k} \uparrow -\rangle = \begin{pmatrix} v_{\boldsymbol{k}} \\ -u_{\boldsymbol{k}} e^{-i\phi_{\boldsymbol{k}}} \end{pmatrix}$	$\begin{pmatrix} 0 \\ -e^{i\theta} \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$\lvert \boldsymbol{k} \downarrow -\rangle = \begin{pmatrix} u_{\boldsymbol{k}} e^{i\phi_{\boldsymbol{k}}} \\ -v_{\boldsymbol{k}} \end{pmatrix}$	$\begin{pmatrix} e^{-i\theta} \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

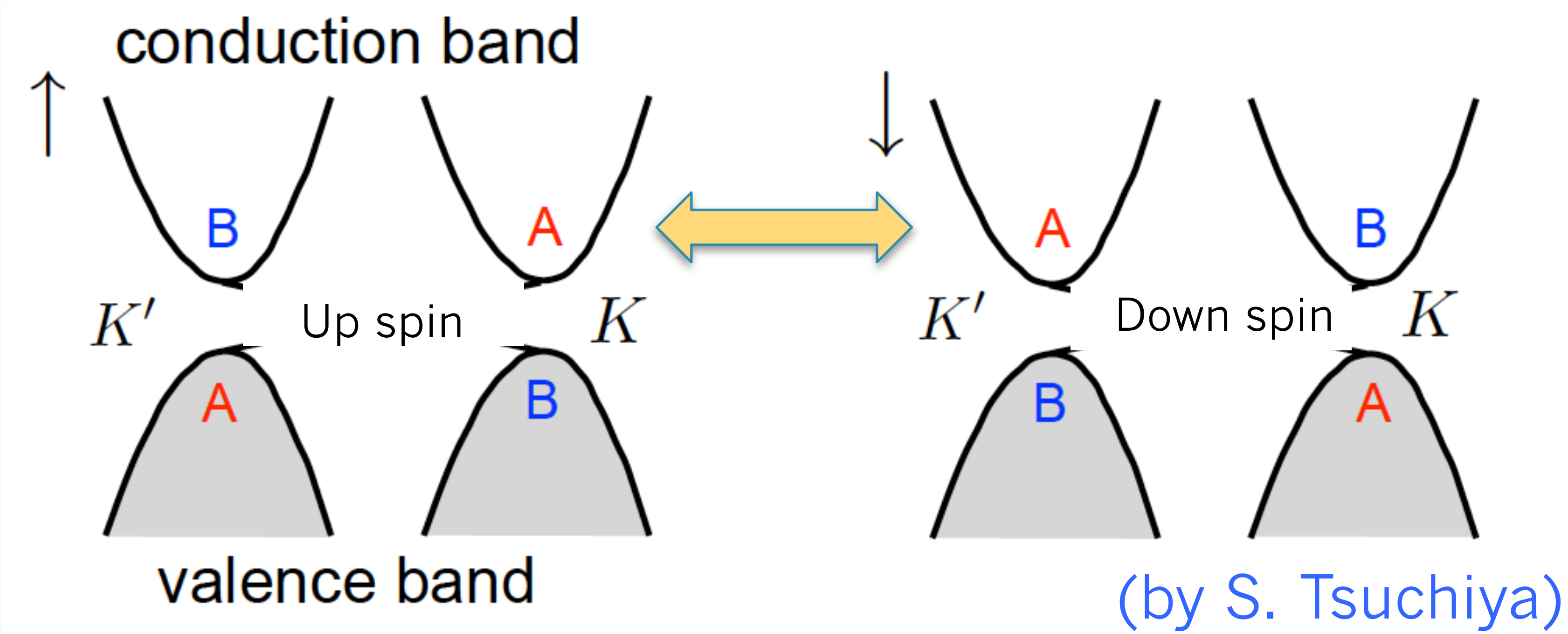
also sublattice eigenstates


conduction band
↑
B
A
K′
Up spin
K
A
B
valence band
↓
A
B
K′
Down spin
K
B
A
(by S. Tsuchiya)



NN (inter-sublattice) attraction
⇒ spin-polarized components of pairing

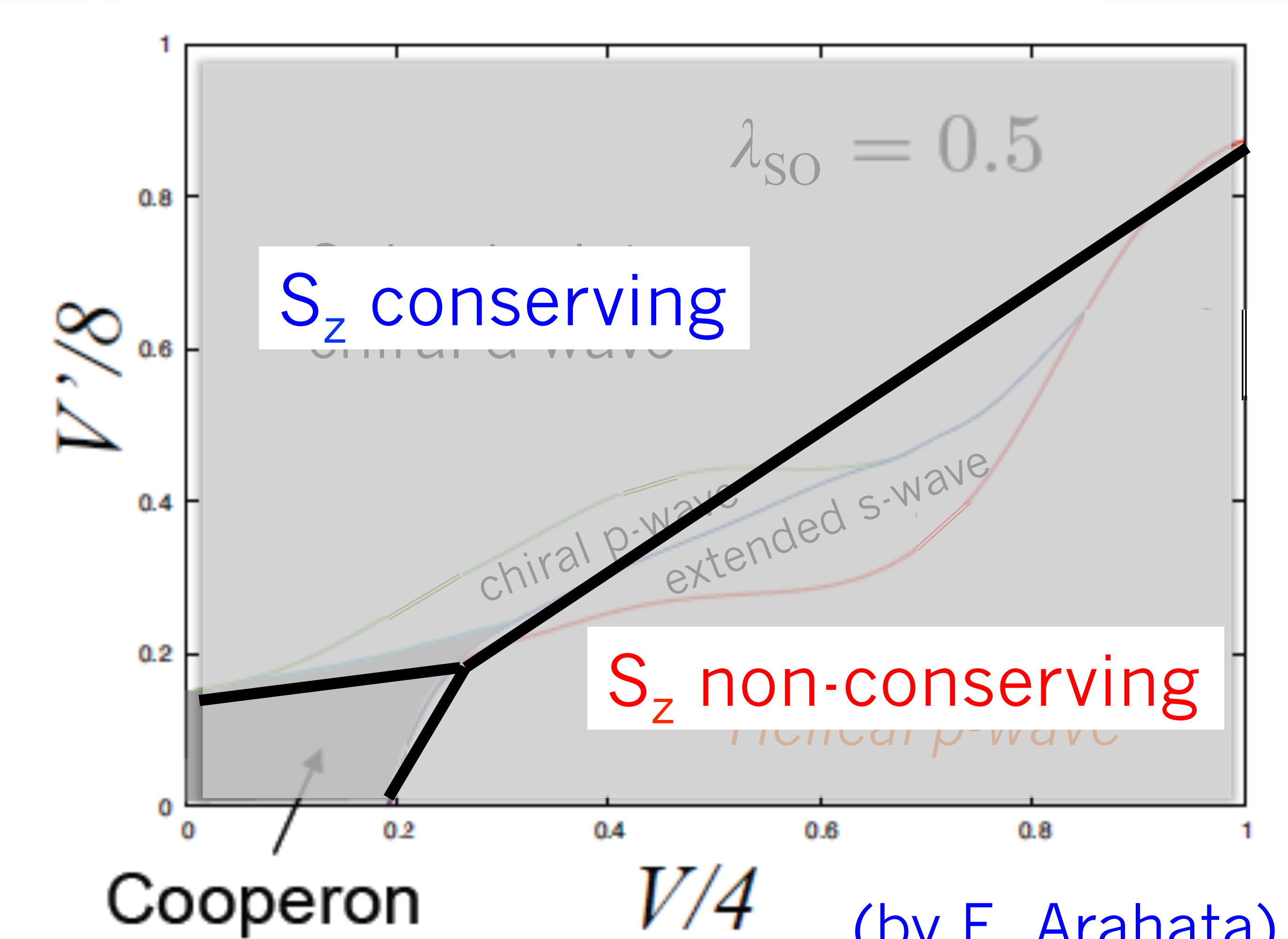
i.e. spin-triplet pairing



NNN (intra-sublattice) attraction
⇒ spin-unpolarized components of pairing

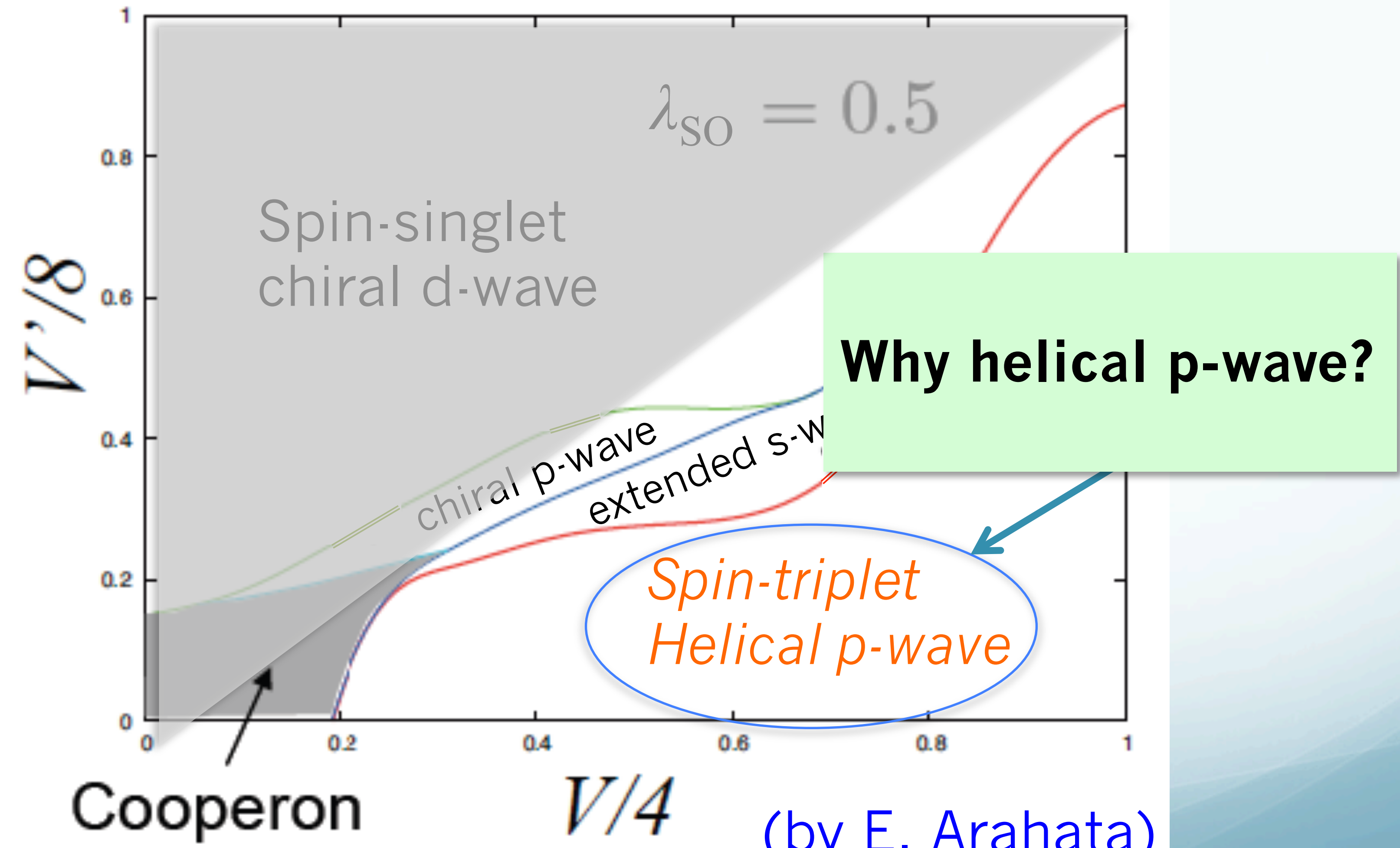
i.e. spin-singlet or S_z-coserving spin-triplet pairing

Analysis of the Gap equation at T=0 with V (*NN*) and V'(*NNN* attraction)



(by E. Arahata)

Analysis of the Gap equation at T=0 with V (*NN*) and V' (*NNN* attraction)



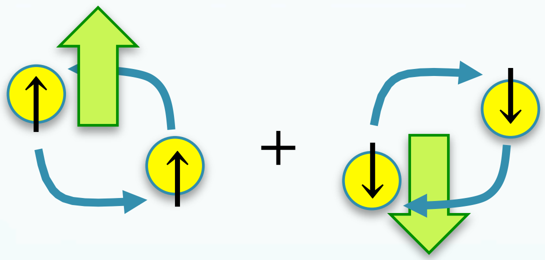
(by E. Arahata)

Helical p-wave state

2D version of superfluid ^{3}He-B

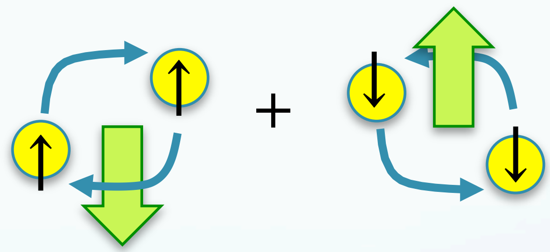
- Superposition of Kramers states
- Two different helicity (parallel and anti-parallel)

Paralell



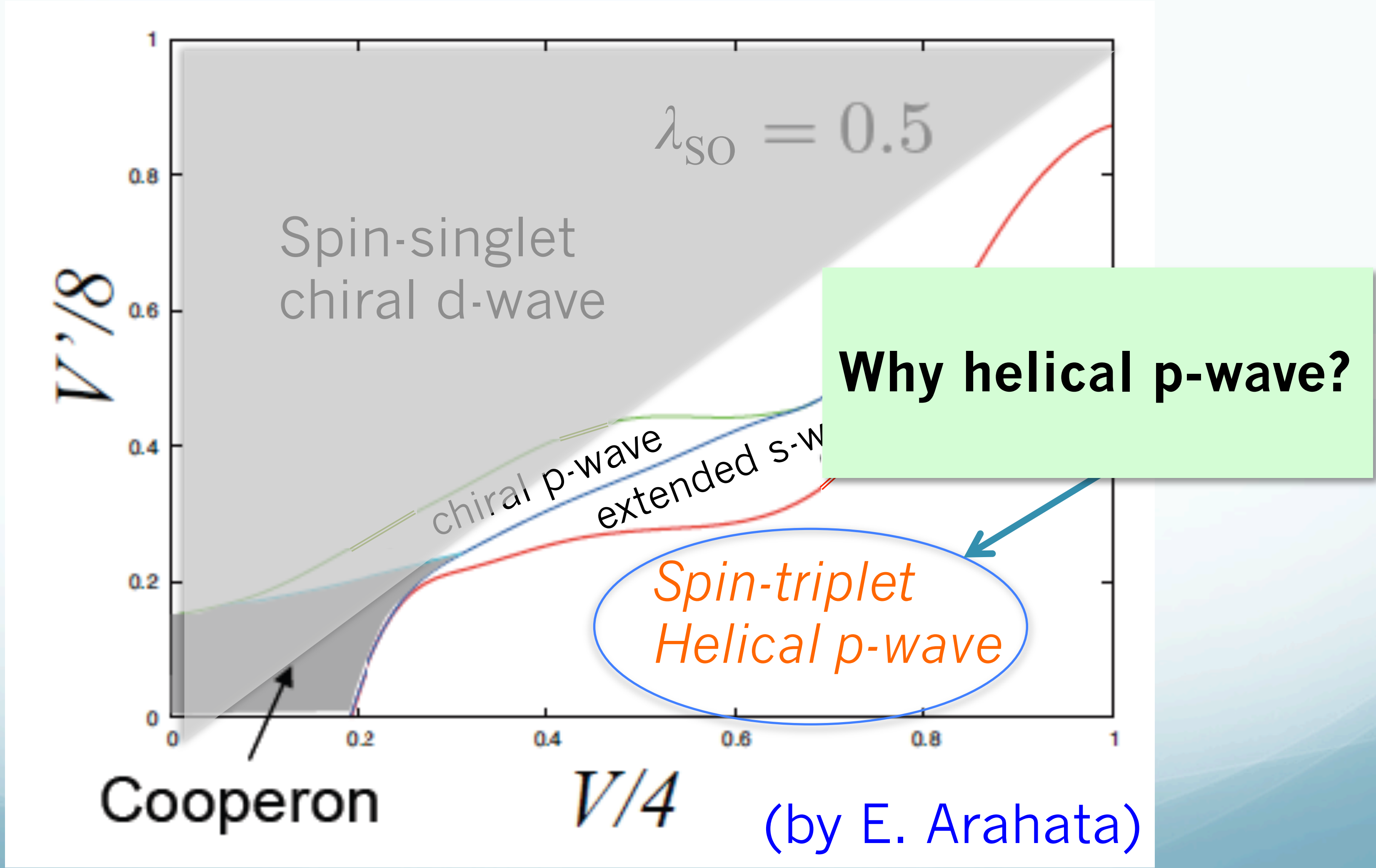
$Lz=1$ $Sz=1$ $Lz=-1$ $Sz=-1$

Anti-parallel



$Lz=-1$ $Sz=1$ $Lz=1$ $Sz=-1$

Analysis of the Gap equation at T=0 with V (*NN*) and V' (*NNN* attraction)



(by E. Arahata)

Diagonalization of H_{KM}

Ann. of conduction electron

Ann. of valence electron

$$\begin{pmatrix} A_{\boldsymbol{k}s} \\ B_{\boldsymbol{k}s} \end{pmatrix} = U(\boldsymbol{k}, s) \begin{pmatrix} \tilde{c}_{\boldsymbol{k}s} \\ \tilde{d}_{\boldsymbol{k}s} \end{pmatrix}$$

$$U(\boldsymbol{k}, \uparrow) = \begin{pmatrix} u_{\boldsymbol{k}} e^{i\phi_{\boldsymbol{k}}} & v_{\boldsymbol{k}} \\ v_{\boldsymbol{k}} & -u_{\boldsymbol{k}} e^{-i\phi_{\boldsymbol{k}}} \end{pmatrix}$$

$$U(\boldsymbol{k}, \downarrow) = \begin{pmatrix} v_{\boldsymbol{k}} & u_{\boldsymbol{k}} e^{i\phi_{\boldsymbol{k}}} \\ u_{\boldsymbol{k}} e^{-i\phi_{\boldsymbol{k}}} & -v_{\boldsymbol{k}} \end{pmatrix}$$

Pairing Hamiltonian for the conduction electrons with NN interaction

$$
\begin{aligned}
H^c &= H^c_{\mathrm{KM}} + H^c_{nn} \\
H^c_{\mathrm{KM}} &= \sum_{\boldsymbol{k}s} \epsilon_{\boldsymbol{k}} \tilde{c}^\dagger_{\boldsymbol{k}s} \tilde{c}_{\boldsymbol{k}s} \\
H^c_{nn} &= -\frac{V}{M} \sum_{\boldsymbol{k},\boldsymbol{p}} t_{\boldsymbol{k}-\boldsymbol{p}} e^{-i(\phi_{\boldsymbol{k}}-\phi_{\boldsymbol{p}})} u^2_{\boldsymbol{k}} u^2_{\boldsymbol{p}} \tilde{c}^\dagger_{\boldsymbol{k}\uparrow} \tilde{c}^\dagger_{-\boldsymbol{k}\uparrow} \tilde{c}_{-\boldsymbol{p}\uparrow} \tilde{c}_{\boldsymbol{p}\uparrow} \\
&\quad -\frac{V}{M} \sum_{\boldsymbol{k},\boldsymbol{p}} t_{\boldsymbol{k}-\boldsymbol{p}} e^{-i(\phi_{\boldsymbol{k}}-\phi_{\boldsymbol{p}})} v^2_{\boldsymbol{k}} v^2_{\boldsymbol{p}} \tilde{c}^\dagger_{\boldsymbol{k}\downarrow} \tilde{c}^\dagger_{-\boldsymbol{k}\downarrow} \tilde{c}_{-\boldsymbol{p}\downarrow} \tilde{c}_{\boldsymbol{p}\downarrow} \\
&\quad -\frac{V}{M} \sum_{\boldsymbol{k},\boldsymbol{p}} t_{\boldsymbol{k}-\boldsymbol{p}} u_{\boldsymbol{k}} v_{\boldsymbol{k}} u_{\boldsymbol{p}} v_{\boldsymbol{p}} \tilde{c}^\dagger_{\boldsymbol{k}\uparrow} \tilde{c}^\dagger_{-\boldsymbol{k}\downarrow} \tilde{c}_{-\boldsymbol{p}\downarrow} \tilde{c}_{\boldsymbol{p}\uparrow} \\
&\quad -\frac{V}{M} \sum_{\boldsymbol{k},\boldsymbol{p}} t_{\boldsymbol{k}-\boldsymbol{p}} u_{\boldsymbol{k}} v_{\boldsymbol{k}} u_{\boldsymbol{p}} v_{\boldsymbol{p}} \tilde{c}^\dagger_{\boldsymbol{k}\downarrow} \tilde{c}^\dagger_{-\boldsymbol{k}\uparrow} \tilde{c}_{-\boldsymbol{p}\uparrow} \tilde{c}_{\boldsymbol{p}\downarrow},
\end{aligned}
$$

M: # of the unit cells

BCS mean field (Gap matrix)

$$\hat{\Delta}(\boldsymbol{k}) = \begin{pmatrix} \Delta_{\uparrow\uparrow}(\boldsymbol{k}) & \Delta_{\uparrow\downarrow}(\boldsymbol{k}) \\ \Delta_{\downarrow\uparrow}(\boldsymbol{k}) & \Delta_{\downarrow\downarrow}(\boldsymbol{k}) \end{pmatrix}$$

$$\Delta_{\uparrow\uparrow}(\boldsymbol{k}) = -\frac{V}{M} u_{\boldsymbol{k}}^2 e^{-i\phi_{\boldsymbol{k}}} \sum_{\boldsymbol{p}} t_{\boldsymbol{k}-\boldsymbol{p}} u_{\boldsymbol{p}}^2 e^{i\phi_{\boldsymbol{p}}} \langle \tilde{c}_{-\boldsymbol{k}'\uparrow} \tilde{c}_{\boldsymbol{p}\uparrow} \rangle$$

$$\Delta_{\downarrow\downarrow}(\boldsymbol{k}) = -\frac{V}{M} v_{\boldsymbol{k}}^2 e^{-i\phi_{\boldsymbol{k}}} \sum_{\boldsymbol{p}} t_{\boldsymbol{k}-\boldsymbol{p}} v_{\boldsymbol{p}}^2 e^{i\phi_{\boldsymbol{p}}} \langle \tilde{c}_{-\boldsymbol{p}\downarrow} \tilde{c}_{\boldsymbol{p}\downarrow} \rangle$$

$$\Delta_{\uparrow\downarrow}(\boldsymbol{k}) = -\frac{V}{M} u_{\boldsymbol{k}} v_{\boldsymbol{k}} \sum_{\boldsymbol{p}} t_{\boldsymbol{k}-\boldsymbol{p}} u_{\boldsymbol{k}'} v_{\boldsymbol{p}} \langle \tilde{c}_{-\boldsymbol{p}\downarrow} \tilde{c}_{\boldsymbol{p}\uparrow} \rangle$$

$$\Delta_{\downarrow\uparrow}(\boldsymbol{k}) = -\frac{V}{M} u_{\boldsymbol{k}} v_{\boldsymbol{k}} \sum_{\boldsymbol{p}} t_{\boldsymbol{k}-\boldsymbol{p}} u_{\boldsymbol{p}} v_{\boldsymbol{p}} \langle \tilde{c}_{-\boldsymbol{p}\uparrow} \tilde{c}_{\boldsymbol{p}\downarrow} \rangle$$

$$\Delta_{\uparrow\uparrow}(\boldsymbol{k}) \;=\; -\frac{V}{M} u_{\boldsymbol{k}}^2 e^{-i\phi_{\boldsymbol{k}}} \sum_{\boldsymbol{p}} t_{\boldsymbol{k}-\boldsymbol{p}} u_{\boldsymbol{p}}^2 e^{i\phi_{\boldsymbol{p}}} \langle \tilde{c}_{-\boldsymbol{p}\uparrow} \tilde{c}_{\boldsymbol{p}\uparrow} \rangle$$

$$\boldsymbol{k}, \boldsymbol{p} \in K, K'$$

would be dominant

$$\begin{aligned}\Delta_{\uparrow\uparrow}(\boldsymbol{k}) &= -\frac{V}{M}u_{\boldsymbol{k}}^2 e^{-i\phi_{\boldsymbol{k}}}\sum_{\boldsymbol{p}} t_{\boldsymbol{k}-\boldsymbol{p}} u_{\boldsymbol{p}}^2 e^{i\phi_{\boldsymbol{p}}}\langle\tilde{c}_{-\boldsymbol{p}\uparrow}\tilde{c}_{\boldsymbol{p}\uparrow}\rangle \\ &\simeq -\frac{V}{M}u_{\boldsymbol{k}}^2 e^{-i\phi_{\boldsymbol{k}}}\left\{\sum_{\boldsymbol{p}\in K}+\sum_{\boldsymbol{p}\in K'}\right\} t_{\boldsymbol{k}-\boldsymbol{p}} u_{\boldsymbol{p}}^2 e^{i\phi_{\boldsymbol{p}}}\langle\tilde{c}_{-\boldsymbol{p}\uparrow}\tilde{c}_{\boldsymbol{p}\uparrow}\rangle\end{aligned}$$

$$\Delta_{\uparrow\uparrow}(\boldsymbol{k}) = -\frac{V}{M} u_{\boldsymbol{k}}^2 e^{-i\phi_{\boldsymbol{k}}} \sum_{\boldsymbol{p}} t_{\boldsymbol{k}-\boldsymbol{p}} u_{\boldsymbol{p}}^2 e^{i\phi_{\boldsymbol{p}}} \langle \tilde{c}_{-\boldsymbol{p}\uparrow} \tilde{c}_{\boldsymbol{p}\uparrow} \rangle$$

$$\simeq -\frac{V}{M} u_{\boldsymbol{k}}^2 e^{-i\phi_{\boldsymbol{k}}} \left\{ \sum_{\boldsymbol{p}\in K} + \sum_{\boldsymbol{p}\in K'} \right\} t_{\boldsymbol{k}-\boldsymbol{p}} u_{\boldsymbol{p}}^2 e^{i\phi_{\boldsymbol{p}}} \langle \tilde{c}_{-\boldsymbol{p}\uparrow} \tilde{c}_{\boldsymbol{p}\uparrow} \rangle$$

	around K	around K'
$\lvert t_{\boldsymbol{k}} \rvert$	0	0
$\phi_{\boldsymbol{k}} = \arg(t_{\boldsymbol{k}})$	$\tan^{-1}\frac{\delta k_y}{\delta k_x} \equiv \theta_{\delta\boldsymbol{k}}$	$-\tan^{-1}\frac{\delta k_y}{\delta k_x} = -\theta_{\delta\boldsymbol{k}}$
$\lambda_{\boldsymbol{k}}$	λ_{SO}	$-\lambda_{\mathrm{SO}}$
$\epsilon_{\boldsymbol{k}}$	$\lvert\lambda_{\mathrm{SO}}\rvert$	$\lvert\lambda_{\mathrm{SO}}\rvert$
	$\cdot\ \lambda_{\mathrm{SO}} > 0$	$\cdot\ \lambda_{\mathrm{SO}} > 0$
$u_{\boldsymbol{k}} = \sqrt{\frac{1}{2}\left(1 + \frac{\lambda_{\boldsymbol{k}}}{\epsilon_{\boldsymbol{k}}}\right)}$	1	0
$v_{\boldsymbol{k}} = \sqrt{\frac{1}{2}\left(1 - \frac{\lambda_{\boldsymbol{k}}}{\epsilon_{\boldsymbol{k}}}\right)}$	0	1

$$\begin{aligned}
\Delta_{\uparrow\uparrow}(\boldsymbol{k}) &= -\frac{V}{M} u_{\boldsymbol{k}}^2 e^{-i\phi_{\boldsymbol{k}}} \sum_{\boldsymbol{p}} t_{\boldsymbol{k}-\boldsymbol{p}} u_{\boldsymbol{p}}^2 e^{i\phi_{\boldsymbol{p}}} \langle \tilde{c}_{-\boldsymbol{p}\uparrow} \tilde{c}_{\boldsymbol{p}\uparrow} \rangle \\
&\simeq -\frac{V}{M} u_{\boldsymbol{k}}^2 e^{-i\phi_{\boldsymbol{k}}} \left\{ \sum_{\boldsymbol{p}\in K} + \sum_{\boldsymbol{p}\in K'} \right\} t_{\boldsymbol{k}-\boldsymbol{p}} u_{\boldsymbol{p}}^2 e^{i\phi_{\boldsymbol{p}}} \langle \tilde{c}_{-\boldsymbol{p}\uparrow} \tilde{c}_{\boldsymbol{p}\uparrow} \rangle \\
&\simeq -\frac{V}{M} u_{\boldsymbol{k}}^2 e^{-i\phi_{\boldsymbol{k}}} \sum_{\boldsymbol{p}\in K} t_{\boldsymbol{k}-\boldsymbol{p}} u_{\boldsymbol{p}}^2 e^{i\phi_{\boldsymbol{p}}} \langle \tilde{c}_{-\boldsymbol{p}\uparrow} \tilde{c}_{\boldsymbol{p}\uparrow} \rangle
\end{aligned}$$

$$\begin{aligned}\Delta_{\uparrow\uparrow}(\boldsymbol{k}) &= -\frac{V}{M}u_{\boldsymbol{k}}^2 e^{-i\phi_{\boldsymbol{k}}}\sum_{\boldsymbol{p}} t_{\boldsymbol{k}-\boldsymbol{p}}u_{\boldsymbol{p}}^2 e^{i\phi_{\boldsymbol{p}}}\langle\tilde{c}_{-\boldsymbol{p}\uparrow}\tilde{c}_{\boldsymbol{p}\uparrow}\rangle \\ &\simeq -\frac{V}{M}u_{\boldsymbol{k}}^2 e^{-i\phi_{\boldsymbol{k}}}\left\{\sum_{\boldsymbol{p}\in K}+\sum_{\boldsymbol{p}\in K'}\right\} t_{\boldsymbol{k}-\boldsymbol{p}}u_{\boldsymbol{p}}^2 e^{i\phi_{\boldsymbol{p}}}\langle\tilde{c}_{-\boldsymbol{p}\uparrow}\tilde{c}_{\boldsymbol{p}\uparrow}\rangle \\ &\simeq -\frac{V}{M}u_{\boldsymbol{k}}^2 e^{-i\phi_{\boldsymbol{k}}}\sum_{\boldsymbol{p}\in K} t_{\boldsymbol{k}-\boldsymbol{p}}u_{\boldsymbol{p}}^2 e^{i\phi_{\boldsymbol{p}}}\langle\tilde{c}_{-\boldsymbol{p}\uparrow}\tilde{c}_{\boldsymbol{p}\uparrow}\rangle\end{aligned}$$

$$\boldsymbol{k},\boldsymbol{p}\in K$$

$$\boldsymbol{t}_{\boldsymbol{k}-\boldsymbol{p}} = \sum_{i=1}^{3} t e^{i(\boldsymbol{k}-\boldsymbol{p})\cdot\boldsymbol{a}_i} \simeq \boldsymbol{t}_{\boldsymbol{k}-\boldsymbol{p}\simeq\boldsymbol{0}} = 3t$$

$$\begin{aligned}
\Delta_{\uparrow\uparrow}(\boldsymbol{k}) &= -\frac{V}{M} u_{\boldsymbol{k}}^2 e^{-i\phi_{\boldsymbol{k}}} \sum_{\boldsymbol{p}} t_{\boldsymbol{k}-\boldsymbol{p}} u_{\boldsymbol{p}}^2 e^{i\phi_{\boldsymbol{p}}} \langle \tilde{c}_{-\boldsymbol{p}\uparrow} \tilde{c}_{\boldsymbol{p}\uparrow} \rangle \\
&\simeq -\frac{V}{M} u_{\boldsymbol{k}}^2 e^{-i\phi_{\boldsymbol{k}}} \left\{ \sum_{\boldsymbol{p}\in K} + \sum_{\boldsymbol{p}\in K'} \right\} t_{\boldsymbol{k}-\boldsymbol{p}} u_{\boldsymbol{p}}^2 e^{i\phi_{\boldsymbol{p}}} \langle \tilde{c}_{-\boldsymbol{p}\uparrow} \tilde{c}_{\boldsymbol{p}\uparrow} \rangle \\
&\simeq -\frac{V}{M} u_{\boldsymbol{k}}^2 e^{-i\phi_{\boldsymbol{k}}} \sum_{\boldsymbol{p}\in K} t_{\boldsymbol{k}-\boldsymbol{p}} u_{\boldsymbol{p}}^2 e^{i\phi_{\boldsymbol{p}}} \langle \tilde{c}_{-\boldsymbol{p}\uparrow} \tilde{c}_{\boldsymbol{p}\uparrow} \rangle \\
&\simeq -\frac{V}{M} u_{\boldsymbol{k}}^2 e^{-i\phi_{\boldsymbol{k}}} \sum_{\boldsymbol{p}\in K} u_{\boldsymbol{p}}^2 e^{i\phi_{\boldsymbol{p}}} \langle \tilde{c}_{-\boldsymbol{p}\uparrow} \tilde{c}_{\boldsymbol{p}\uparrow} \rangle
\end{aligned}$$

$$
\begin{aligned}
\Delta_{\uparrow\uparrow}(\boldsymbol{k}) &= -\frac{V}{M} u_{\boldsymbol{k}}^2 e^{-i\phi_{\boldsymbol{k}}} \sum_{\boldsymbol{p}} t_{\boldsymbol{k}-\boldsymbol{p}} u_{\boldsymbol{p}}^2 e^{i\phi_{\boldsymbol{p}}} \langle \tilde{c}_{-\boldsymbol{p}\uparrow} \tilde{c}_{\boldsymbol{p}\uparrow} \rangle \\
&\simeq -\frac{V}{M} u_{\boldsymbol{k}}^2 e^{-i\phi_{\boldsymbol{k}}} \left\{ \sum_{\boldsymbol{p}\in K} + \sum_{\boldsymbol{p}\in K'} \right\} t_{\boldsymbol{k}-\boldsymbol{p}} u_{\boldsymbol{p}}^2 e^{i\phi_{\boldsymbol{p}}} \langle \tilde{c}_{-\boldsymbol{p}\uparrow} \tilde{c}_{\boldsymbol{p}\uparrow} \rangle \\
&\simeq -\frac{V}{M} u_{\boldsymbol{k}}^2 e^{-i\phi_{\boldsymbol{k}}} \sum_{\boldsymbol{p}\in K} t_{\boldsymbol{k}-\boldsymbol{p}} u_{\boldsymbol{p}}^2 e^{i\phi_{\boldsymbol{p}}} \langle \tilde{c}_{-\boldsymbol{p}\uparrow} \tilde{c}_{\boldsymbol{p}\uparrow} \rangle \\
&\simeq -\frac{V}{M} u_{\boldsymbol{k}}^2 e^{-i\phi_{\boldsymbol{k}}} \sum_{\boldsymbol{p}\in K} u_{\boldsymbol{p}}^2 e^{i\phi_{\boldsymbol{p}}} \langle \tilde{c}_{-\boldsymbol{p}\uparrow} \tilde{c}_{\boldsymbol{p}\uparrow} \rangle \\
&= \Delta u_{\boldsymbol{k}}^2 e^{-i\phi_{\boldsymbol{k}}}
\end{aligned}
$$

$$\Delta_{\uparrow\uparrow}(\boldsymbol{k}) \;=$$

$$\simeq$$

$$\simeq$$

$$\simeq$$

	around K	around K'
$\lvert t_{\boldsymbol{k}}\rvert$	0	0
$\phi_{\boldsymbol{k}} = \arg(t_{\boldsymbol{k}})$	$\tan^{-1}\frac{\delta k_y}{\delta k_x} \equiv \theta_{\delta\boldsymbol{k}}$	$-\tan^{-1}\frac{\delta k_y}{\delta k_x} = -\theta_{\delta\boldsymbol{k}}$
$\lambda_{\boldsymbol{k}}$	λ_{SO}	$-\lambda_{\mathrm{SO}}$
$\epsilon_{\boldsymbol{k}}$	$\lvert\lambda_{\mathrm{SO}}\rvert$	$\lvert\lambda_{\mathrm{SO}}\rvert$
	$\cdot\ \lambda_{\mathrm{SO}} > 0$	$\cdot\ \lambda_{\mathrm{SO}} > 0$
$u_{\boldsymbol{k}} = \sqrt{\frac{1}{2}\left(1+\frac{\lambda_{\boldsymbol{k}}}{\epsilon_{\boldsymbol{k}}}\right)}$	1	0
$v_{\boldsymbol{k}} = \sqrt{\frac{1}{2}\left(1-\frac{\lambda_{\boldsymbol{k}}}{\epsilon_{\boldsymbol{k}}}\right)}$	0	1

$$= \Delta u_{\boldsymbol{k}}^2 e^{-i\phi_{\boldsymbol{k}}}$$

$$\Delta_{\uparrow\uparrow}(\boldsymbol{k}) \to \begin{cases} \Delta e^{-i\theta_{\delta\boldsymbol{k}}} & (\boldsymbol{k} \in K) \\ 0 & (\boldsymbol{k} \in K') \end{cases}$$

	around K	around K'
$\lvert t_{\boldsymbol{k}} \rvert$	0	0
$\phi_{\boldsymbol{k}} = \arg(t_{\boldsymbol{k}})$	$\tan^{-1}\frac{\delta k_y}{\delta k_x} \equiv \theta_{\delta \boldsymbol{k}}$	$-\tan^{-1}\frac{\delta k_y}{\delta k_x} = -\theta_{\delta \boldsymbol{k}}$
$\lambda_{\boldsymbol{k}}$	λ_{SO}	$-\lambda_{\mathrm{SO}}$
$\epsilon_{\boldsymbol{k}}$	$\lvert \lambda_{\mathrm{SO}} \rvert$	$\lvert \lambda_{\mathrm{SO}} \rvert$
	$\cdot\ \lambda_{\mathrm{SO}} > 0$	$\cdot\ \lambda_{\mathrm{SO}} > 0$
$u_{\boldsymbol{k}} = \sqrt{\frac{1}{2}\left(1 + \frac{\lambda_{\boldsymbol{k}}}{\epsilon_{\boldsymbol{k}}}\right)}$	1	0
$v_{\boldsymbol{k}} = \sqrt{\frac{1}{2}\left(1 - \frac{\lambda_{\boldsymbol{k}}}{\epsilon_{\boldsymbol{k}}}\right)}$	0	1
$\Delta_{\uparrow\uparrow}(\boldsymbol{k}) \propto u_{\boldsymbol{k}}^2 e^{-i\phi_{\boldsymbol{k}}}$	$\Delta e^{-i\theta_{\delta \boldsymbol{k}}}$	0
$\Delta_{\downarrow\downarrow}(\boldsymbol{k}) \propto v_{\boldsymbol{k}}^2 e^{-i\phi_{\boldsymbol{k}}}$	0	$\Delta e^{i\theta_{\delta \boldsymbol{k}}}$
$\Delta_{\uparrow\downarrow}(\boldsymbol{k}) \propto u_{\boldsymbol{k}} v_{\boldsymbol{k}}$	0	0
$\Delta_{\downarrow\uparrow}(\boldsymbol{k}) \propto u_{\boldsymbol{k}} v_{\boldsymbol{k}}$	0	0

	around K		around K'	
$\|t_{\boldsymbol{k}}\|$	0		0	
$\phi_{\boldsymbol{k}} = \arg(t_{\boldsymbol{k}})$	$\tan^{-1}\frac{\delta k_y}{\delta k_x} \equiv \theta_{\delta \boldsymbol{k}}$		$-\tan^{-1}\frac{\delta k_y}{\delta k_x} = -\theta_{\delta \boldsymbol{k}}$	
$\lambda_{\boldsymbol{k}}$	λ_{SO}		$-\lambda_{\mathrm{SO}}$	
$\epsilon_{\boldsymbol{k}}$	$\|\lambda_{\mathrm{SO}}\|$		$\|\lambda_{\mathrm{SO}}\|$	
	$\cdot\ \lambda_{\mathrm{SO}} > 0$	$\cdot\ \lambda_{\mathrm{SO}} < 0$	$\cdot\ \lambda_{\mathrm{SO}} > 0$	$\cdot\ \lambda_{\mathrm{SO}} < 0$
$u_{\boldsymbol{k}} = \sqrt{\frac{1}{2}\left(1 + \frac{\lambda_{\boldsymbol{k}}}{\epsilon_{\boldsymbol{k}}}\right)}$	1	0	0	1
$v_{\boldsymbol{k}} = \sqrt{\frac{1}{2}\left(1 - \frac{\lambda_{\boldsymbol{k}}}{\epsilon_{\boldsymbol{k}}}\right)}$	0	1	1	0
$\Delta_{\uparrow\uparrow}(\boldsymbol{k}) \propto u_{\boldsymbol{k}}^2 e^{-i\phi_{\boldsymbol{k}}}$	$\Delta e^{-i\theta_{\delta \boldsymbol{k}}}$	0	0	$\Delta e^{i\theta_{\delta \boldsymbol{k}}}$
$\Delta_{\downarrow\downarrow}(\boldsymbol{k}) \propto v_{\boldsymbol{k}}^2 e^{-i\phi_{\boldsymbol{k}}}$	0	$\Delta e^{-i\theta_{\delta \boldsymbol{k}}}$	$\Delta e^{i\theta_{\delta \boldsymbol{k}}}$	0
$\Delta_{\uparrow\downarrow}(\boldsymbol{k}) \propto u_{\boldsymbol{k}} v_{\boldsymbol{k}}$	0	0	0	0
$\Delta_{\downarrow\uparrow}(\boldsymbol{k}) \propto u_{\boldsymbol{k}} v_{\boldsymbol{k}}$	0	0	0	0

$\lambda_{SO}>0$

$$e^{-i\theta_{\delta \boldsymbol{k}}} \left|\uparrow\uparrow\right\rangle + e^{i\theta_{\delta \boldsymbol{k}}} \left|\downarrow\downarrow\right\rangle$$

winding -2π
S_z= +1

winding +2π
S_z= -1

(anti-parallel helicity)

$\lambda_{SO}<0$

$$e^{i\theta_{\delta \boldsymbol{k}}} \left|\uparrow\uparrow\right\rangle + e^{-i\theta_{\delta \boldsymbol{k}}} \left|\downarrow\downarrow\right\rangle$$

(parallel helicity)

winding 2π
S_z= +1

winding -2π
S_z= -1

- **similar to the helical p-wave state, and would induce it**
- **helicity depends on sgn λ_{SO} (=spin Chern # of ins. phase)**

Summary

The role of the KM spin-orbit coupling for the Cooperon condensation

- S_z-conserving state is stabilized by NNN (intra-sublattice)int.
- S_z-non-conserving helical p-wave state is stabilized by NN (inter-sublattice) int.
- Helicity of helical p-wave state would be related to the band topology (i.e. spin Chern #) induced by the SOC