Tensor Renormalization of Quantum Spin Liquid States

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Outline

1. **Brief introduction to the tensor renormalization**

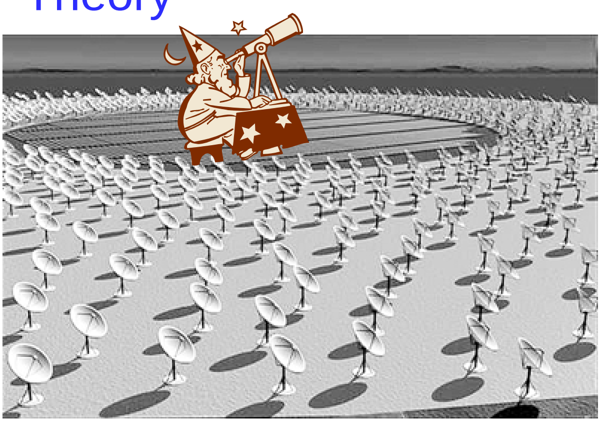
2. **S=1/2 Heisenberg model on the Kagome lattice**
   - ✓ **Tensor network representation of ground state wavefunctions**
   - ✓ **Preliminary results on the ground state energy of the Kagome Heisenberg model**

Tensor Renormalization: A new method to solve quantum many-body problems

Strongly Correlated Systems

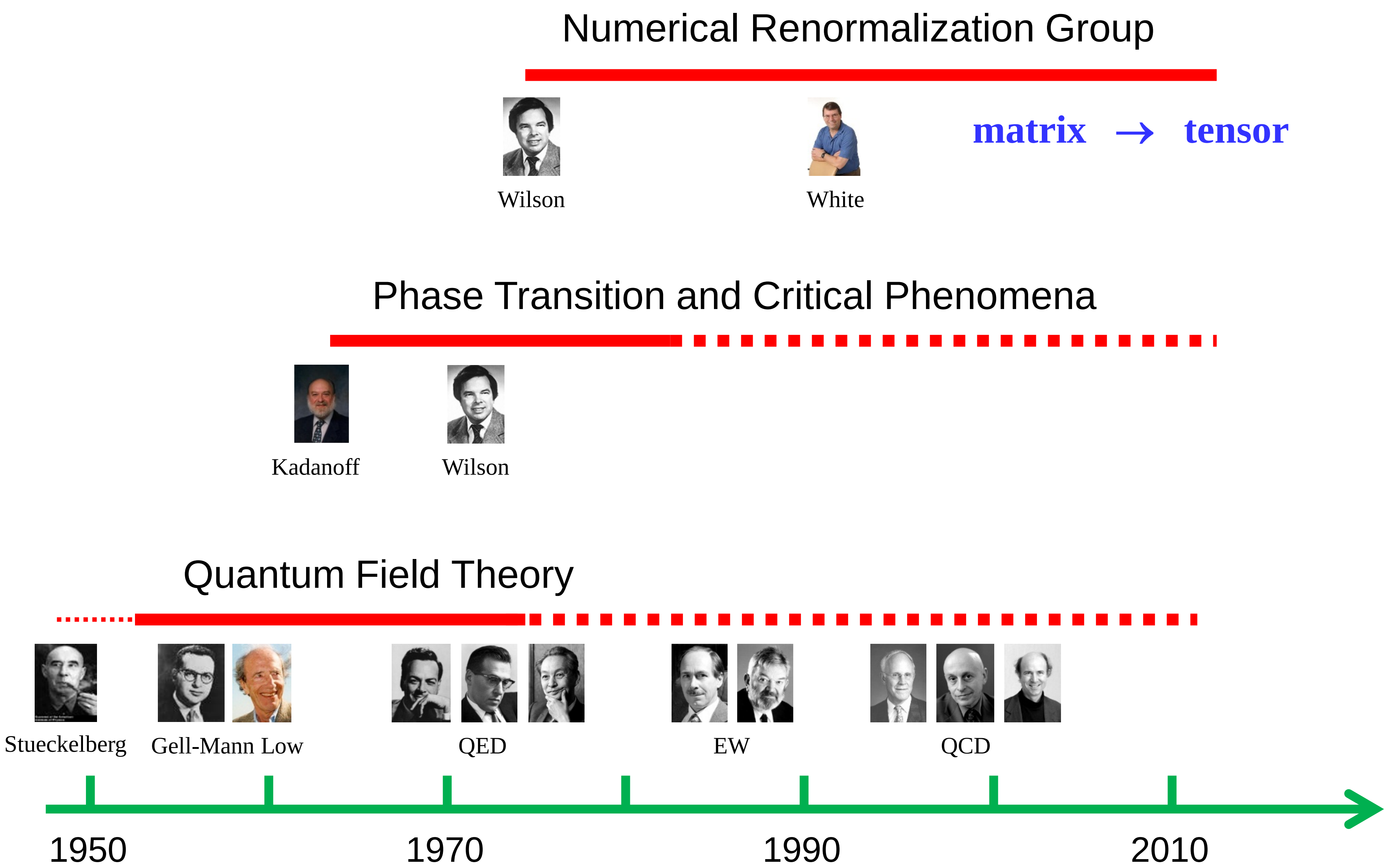
- **To represent the "elephant" using the tensor-network state**

Theory



- **To determine and detect this wave function using tensor renormalization**

Road Map of Renormalization Group Theory



Numerical Renormalization Group

Wilson NRG 1975 -

0 Dimensional problems (single impurity Kondo model)

K. Wilson

Density-matrix Renormalization 1992 -

most powerful method for 1D quantum lattice models

S R White

Renormalization of Tensor Network States

2D or higher dimensional quantum/classical models

No minus sign problem & lattice size can be infinite

Tensor Network States Are Ubiquitous

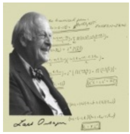
1. **All classical and quantum lattice models are or can be represented as tensor network models**

$$Z = Tr \prod_i T_{x_i x'_i y_i y'_i}$$

2. **Ground state wave functions of quantum lattice models can be represented as tensor-network states**

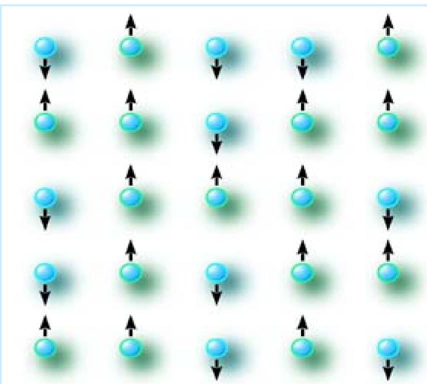
$$|\Psi\rangle = Tr \prod T_{x_i x'_i y_i y'_i}[m_i]|m_i\rangle$$

Example: Tensor Representation of Ising model



Lars Onsager

$$H = -J\sum_{\langle ij\rangle}\sigma_i^z\sigma_j^z$$

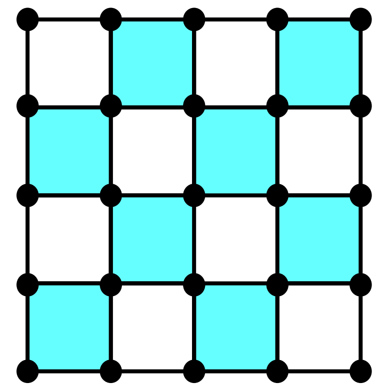


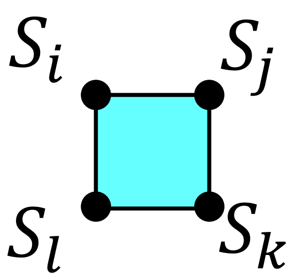
$$\sigma_i^z = -1,\ 1$$

$$Z = \mathrm{Tr}\exp(-\beta H)$$

$$= \mathrm{Tr}\prod_{\blacksquare}\exp(-\beta H_{\blacksquare})$$

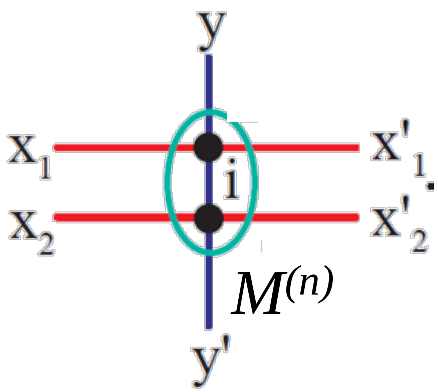
$$= \mathrm{Tr}\prod_{\{S\}} T_{S_iS_jS_kS_l}$$





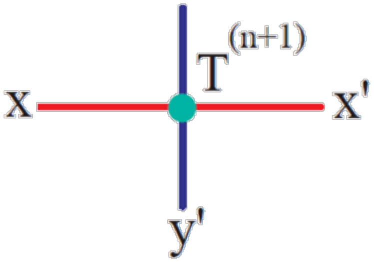
$$= T_{S_iS_jS_kS_l} = \exp(-\beta H_{\blacksquare})$$

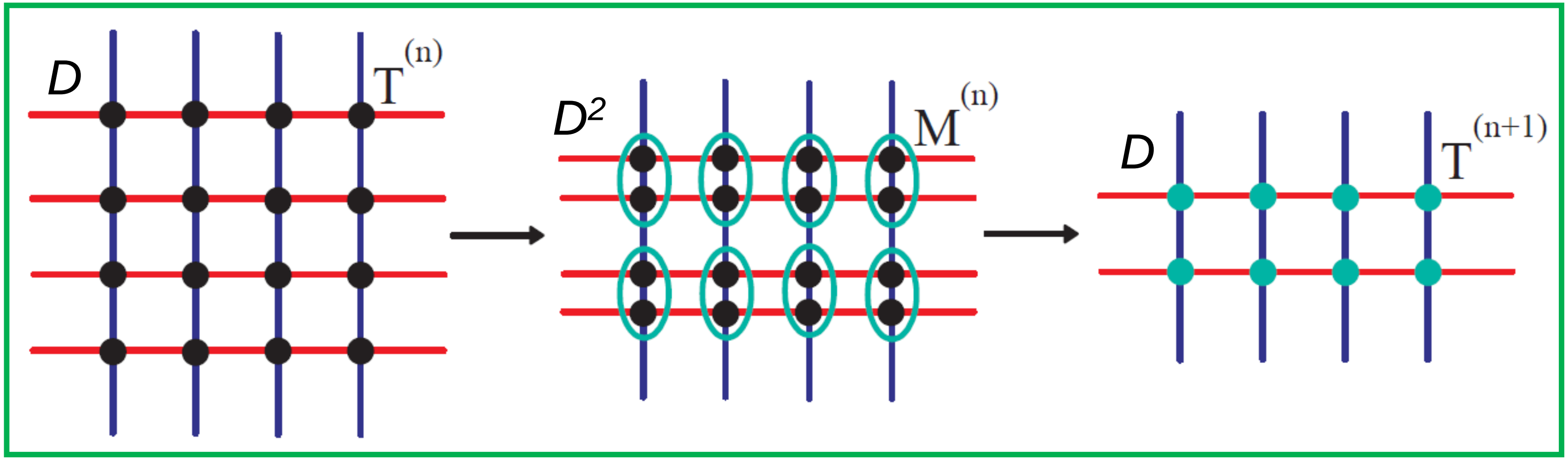
How to renormalize tensor: HOTRG



Higher-order singular value decomposition

Lower-rank approximation

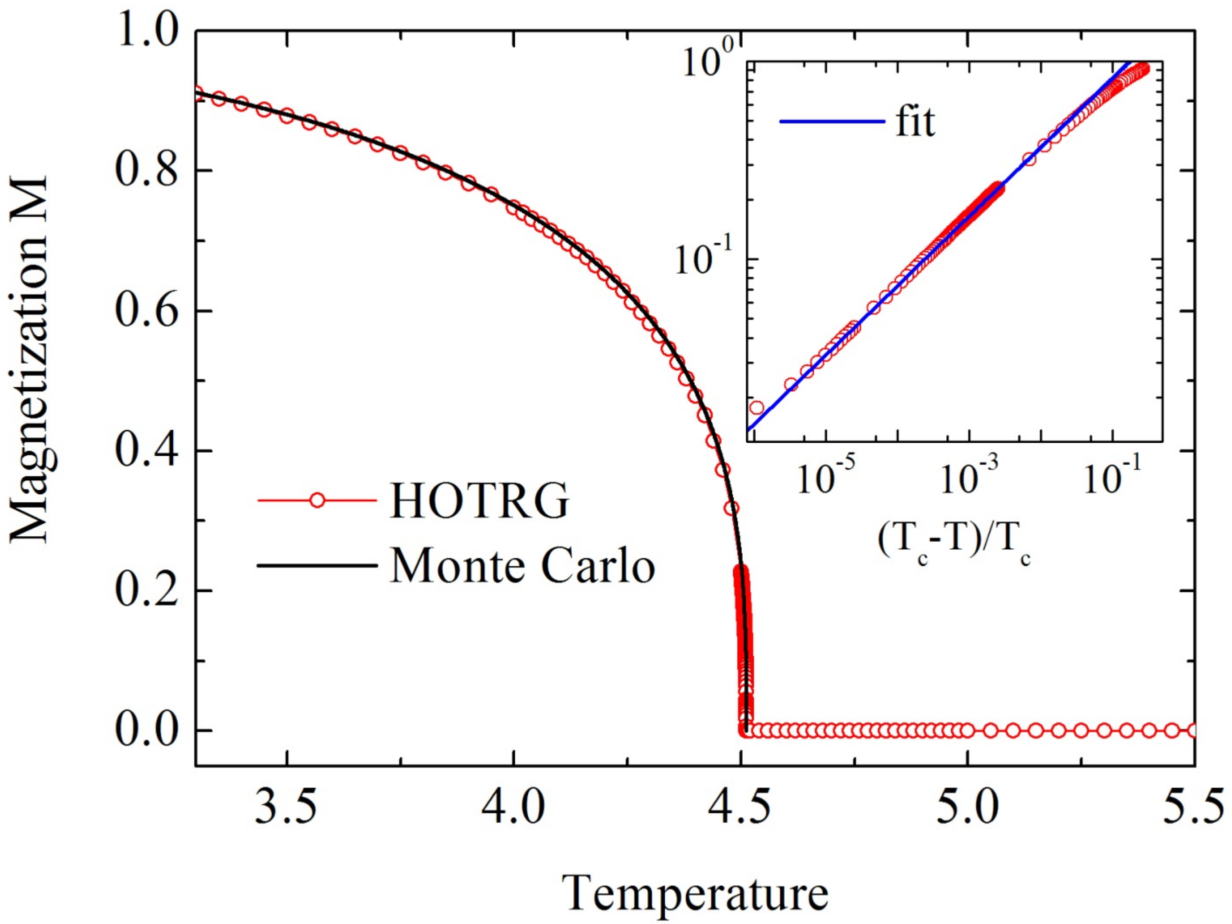




Scaling transformation

Magnetization of 3D Ising model

Xie et al, PRB 86,045139 (2012)



$$M \sim t^{\beta}$$

HOTRG (D=14): 0.3295

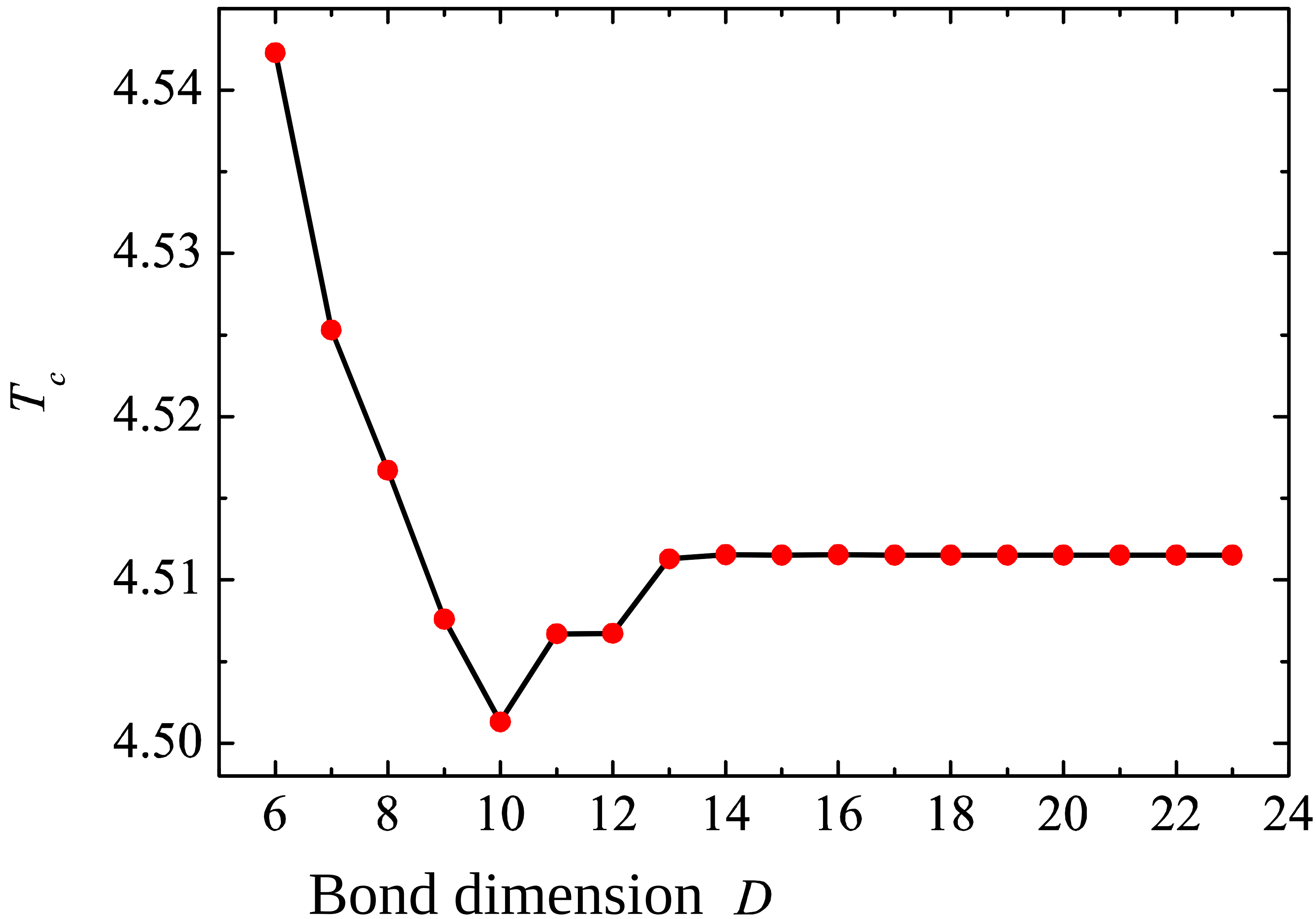
Monte Carlo: 0.3262

Series Expansion: 0.3265

Relative difference is less than 10^{-5}

MC data: A. L. Talapov, H. W. J. Blote, J. Phys. A: Math. Gen. 29, 5727 (1996).

Critical Temperature of 3D Ising model

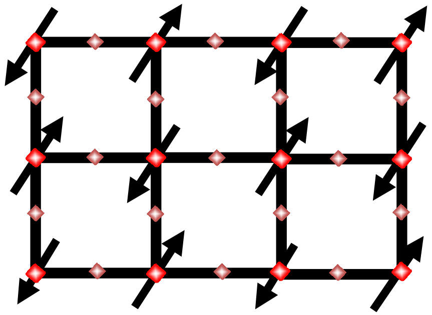


Critical Temperature of 3D Ising model

method	year	T_c
HOTRG D = 16	2012	4.511544
D = 23	2013	4.51152469(1)
NRG of Nishino et al	2005	4.55(4)
Monte Carlo Simulation	2010	4.5115232(17)
	2003	4.5115248(6)
	1996	4.511516
High-temperature expansion	2000	4.511536

Application: search for quantum spin liquid states

Is there a Mott insulator without AFM order --- spin liquid state?



What is Mott insulator?

Mott Picture

One electron per unit cell. Charge gap is due to correlation. AFM is secondary effect.

It is not a band insulator

Slater Picture

Antiferromagnetic ground state. Unit cell is doubled. Then there are 2 electrons per unit cell

It is a band insulator

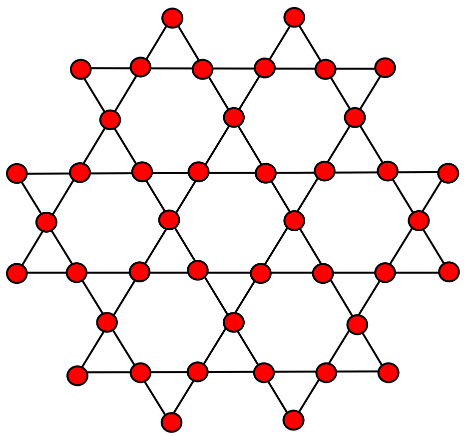
JC Slater, PR **82**, 538 (1951)

Routes to Spin Liquid States

Requirements:

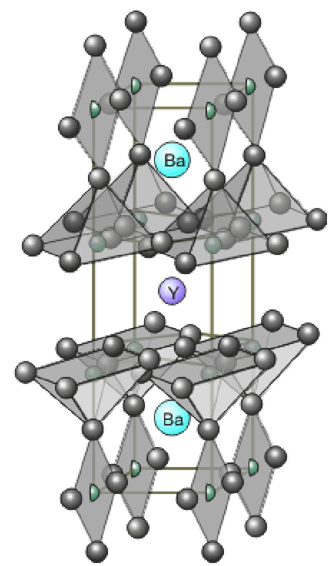
- ✓ **insulator**
- ✓ **odd number of electrons per unit cell**
- ✓ **no long range order**

Route I
Geometrical frustration



Heisenberg model on Kagome

Route II
Proximity to Mott transition



High-Tc cuprates

S=1/2 Kagome Heisenberg model

Valence bond crystal

J. B. Marston and C. Zeng, J Appl Phys **69**, 5962 (1991)
P. Nikolic and T. Senthil, PRB **68**, 214415 (2003)
R. R. P. Singh and D. A. Huse, PRB **76**, 180407 (2007) cluster expansion
R. R. P. Singh and D. A. Huse, PRB **77**, 144415 (2008) cluster expansion
G. Evenbly and G. Vidal, PRL **104**, 187203 (2010) MERA
Y. Iqbal, F. Becca, and D. Poilblanc, PRB **83**, 100404 (2011) variational MC

Z2 quantum spin liquid

S. Sachdev, PRB **45**, 12377 (1992) Large N expansion
H. C. Jiang, Z. Y. Weng, and D. N. Sheng, PRL **101**, 117203 (2008) DMRG
S. Yan, D. A. Huse, and S. R. White, Science **332**, 1173 (2011) DMRG
Depenbrock,McCulloch,Schollwock, PRL **109**, 067201 (2012) DMRG
D. Poilblanc, N. Schuch, D. Perez-Garcia, and J. I. Cirac, PRB **86**, 014404 (2012)
D. Poilblanc and N. Schuch, PRB **87**, 140407 (2013) variational

Gapless quantum spin liquid

Y. Ran, M. Hermele, P. A. Lee, and X.-G. Wen, PRL **98**, 117205 (2007) variational
Y. Iqbal, F. Becca, S. Sorella, and D. Poilblanc, PRB 87, 060405 (2013) variational

Difficulty in the study of quantum spin liquid

Experiment: difficult to rule out all possibilities of orders

Theory: Lack of good analytic/numerical methods

- ✓ **Mean field or variational approach:**

  need accurate guess of the wavefunction

- ✓ **Quantum Monte Carlo:**

  minus sign problem

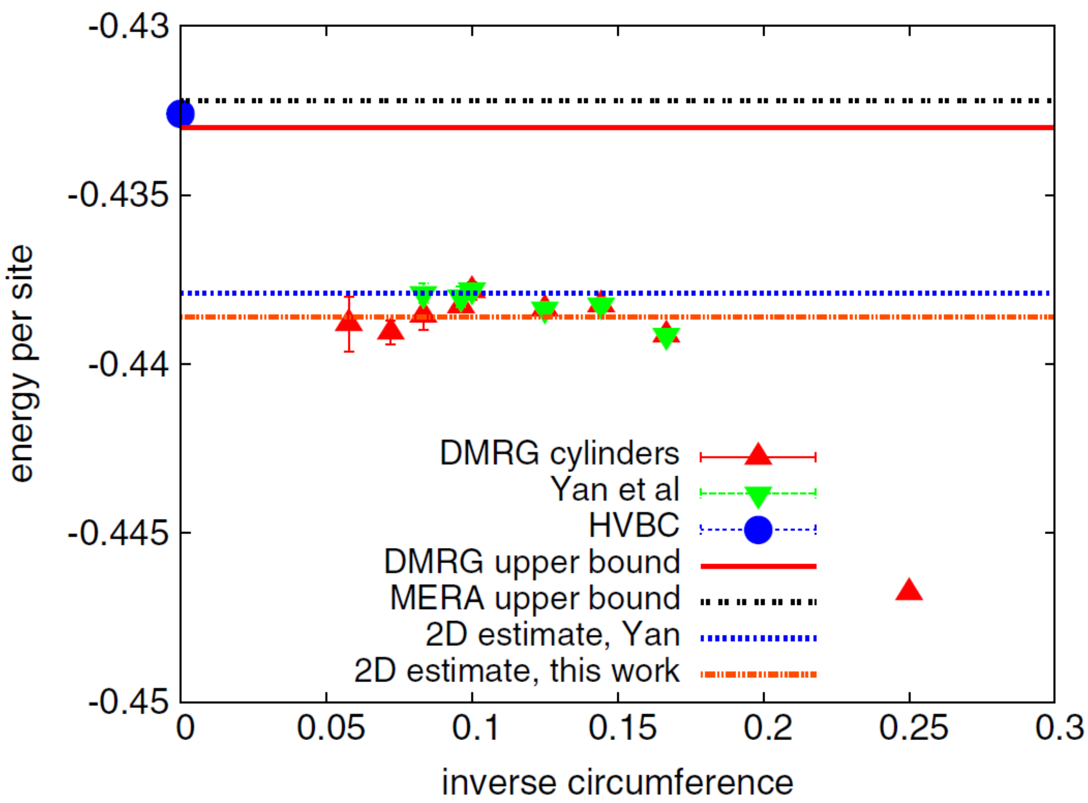
- ✓ **Density Matrix Renormalization Group (DMRG):**

  finite size effect

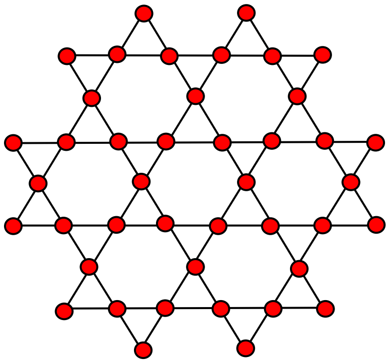
  the number of states need to be retained grows exponentially

  with the circumference (area law of entanglement entropy)

Ground state energy of the S=1/2 Kagome Heisenberg model



$$H = J\sum_{i=1}^{N}\vec{S}_i \cdot \vec{S}_{i+1}$$



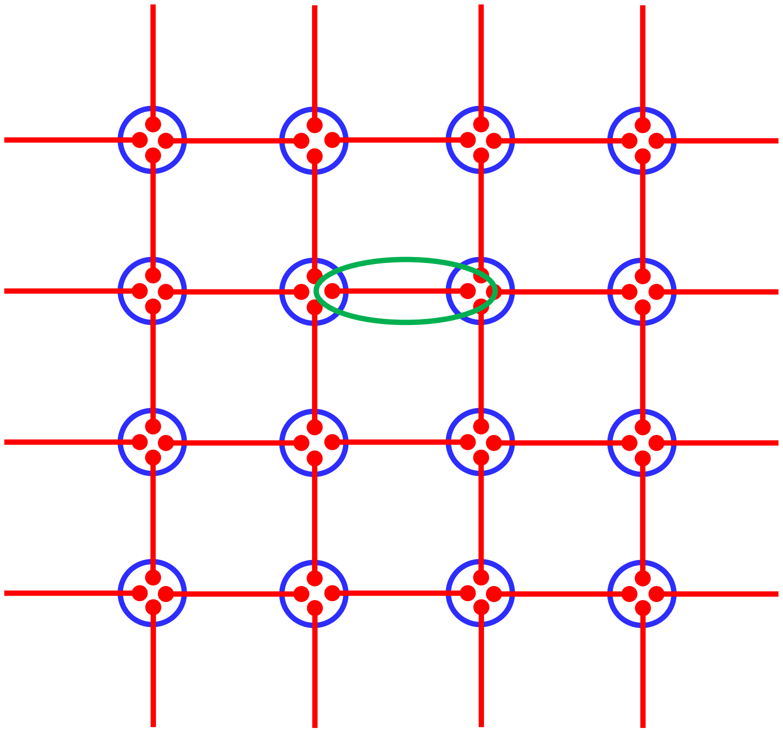
Depenbrock et al, PRL **109**, 067201 (2012)

Ground state energy obtained with different methods

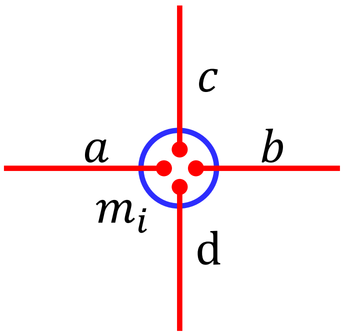
Projected Entangled Pair State (PEPS)

$$|\Psi\rangle = Tr \prod T_{x_i x_i' y_i y_i'}[m_i] | m_i \rangle$$

Virtual basis state   Physical state



$T_{abcd}\,[m_i] =$



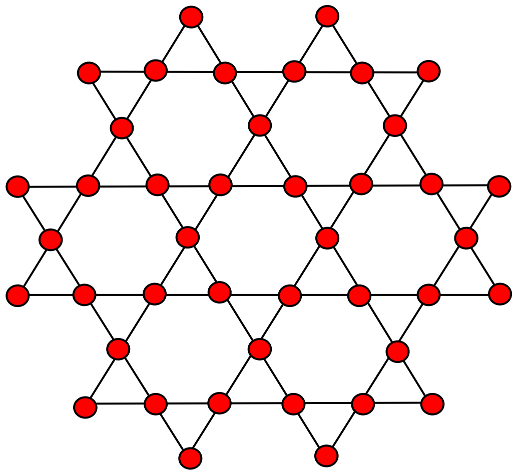
Takes account the pair entanglement accurately

Projected Entangled Pair State (PEPS)

$$|\Psi\rangle = Tr \prod T_{x_i x'_i y_i y'_i}[m_i]|m_i\rangle$$

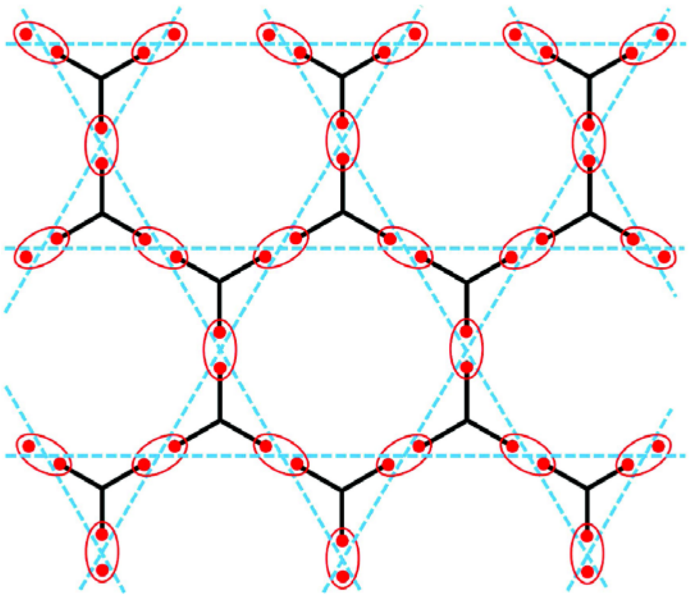
Virtual basis state — Physical state

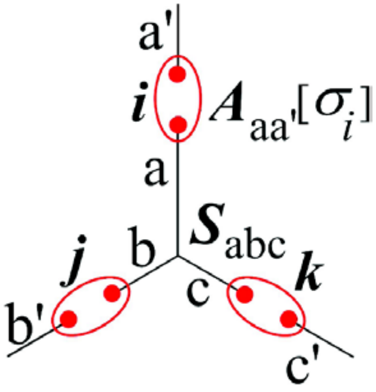
- Successfully applied to the quantum spin models on honeycomb and square lattices
- But, difficult to obtain a converged result if applied to the AFM Heisenberg on the Kagome or other frustrated lattices



Kagome Lattice

Projected Entangled Simplex States (PESS)





Projection tensor

Simplex tensor

- **Virtual spins at each simplex (here triangle), instead of at each pair, form a maximally entangled state**
- **Remove the geometry frustration: The PESS wavefunction on the Kagome lattice is defined on the decorated honeycomb lattice (no frustration)**

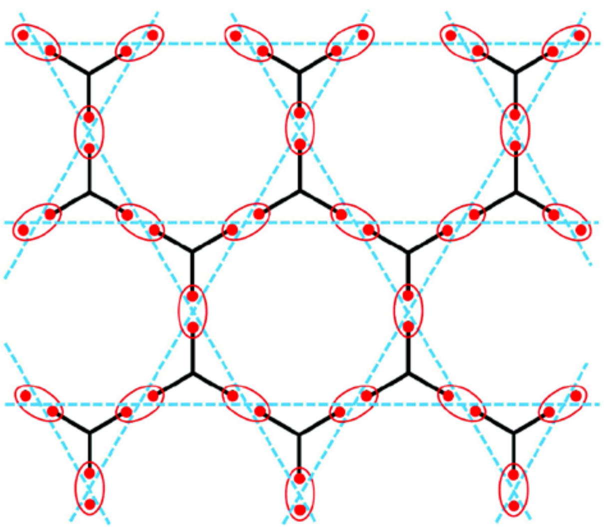
Simplex Solid States

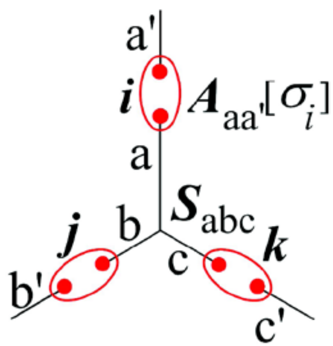
D. P. Arovas, Phys. Rev. B **77**, 104404 (2008)

Example: **S = 2** spin model on the Kagome lattice

A **S = 2** spin is a symmetric superposition of two virtual **S = 1** spins

Three virtual spins at each triangle form a spin singlet

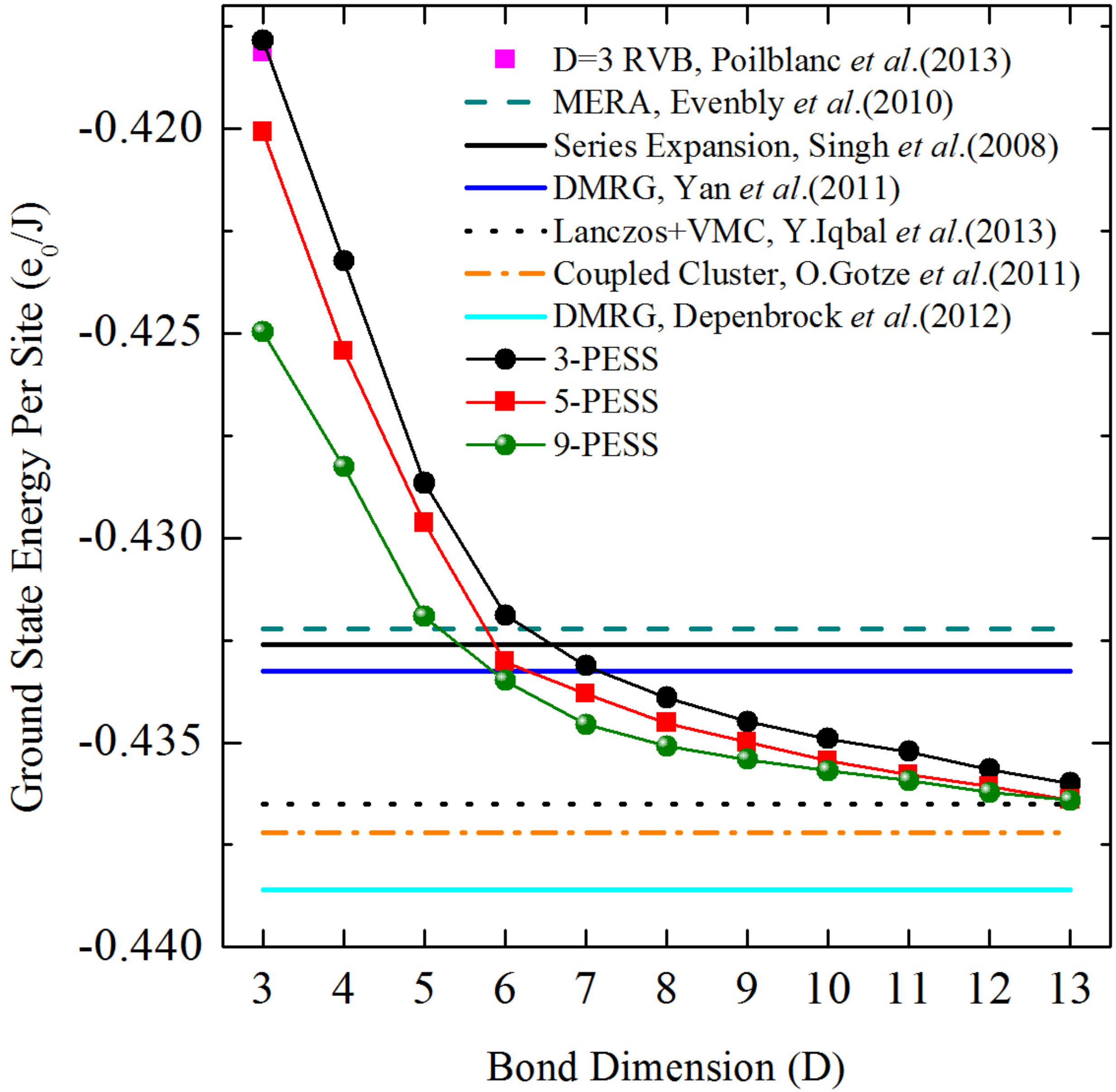




Projection tensor

Simplex tensor

Ground state energy of the S=1/2 Kagome Heisenberg model



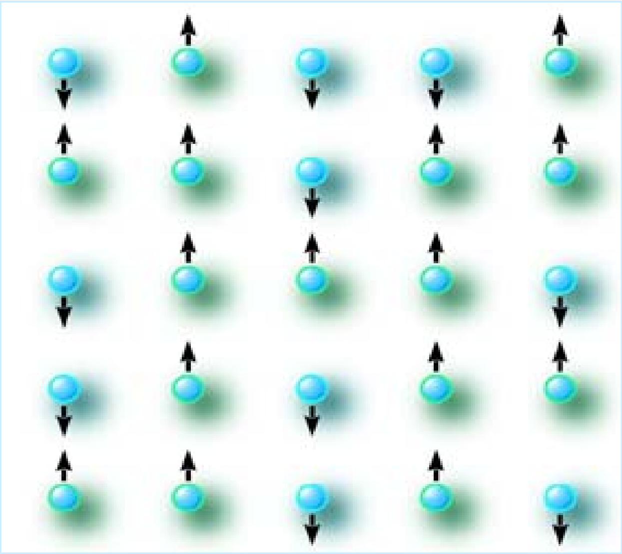
Summary

- **Tensor renormalization provides a powerful tool for studying classical/quantum lattice models**

- **Tensor renormalization with the PESS wave function provides a good framework for studying quantum spin liquid states**

- **But there are still many problems need to be solved**

Is the phase space of quantum many-body system compressible?

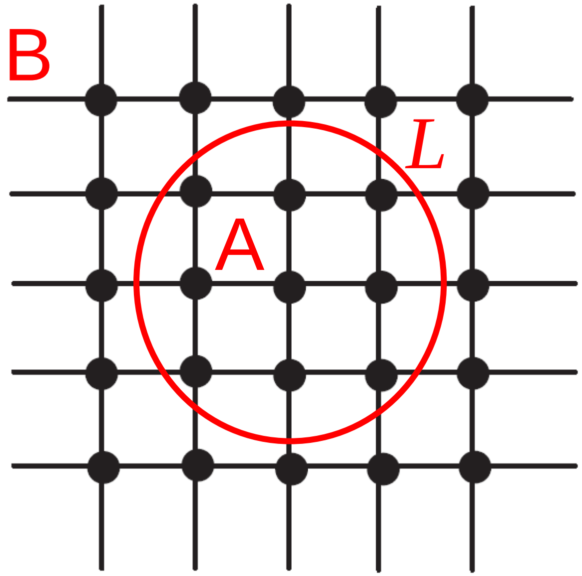


$$N_{total} = 2^{L^2}$$

$$|\psi\rangle = \sum_{k=1}^{N \ll N_{total}} a_k \, |k\rangle$$

basis states

Area Law of Entanglement Entropy



Entanglement between A and B

$$S \propto L \propto \ln N$$

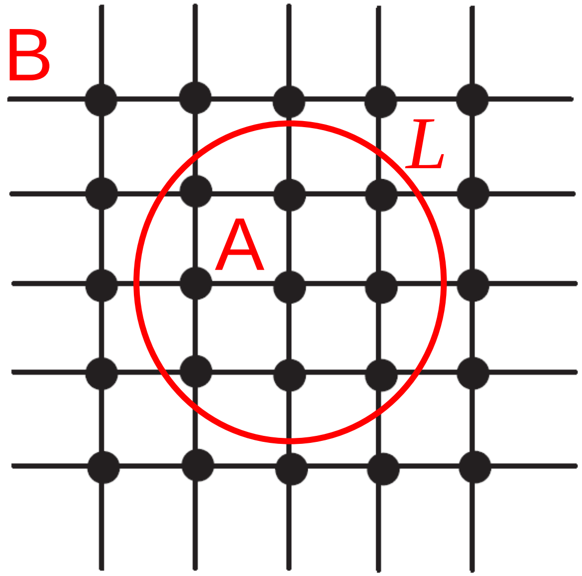
$$N \sim 2^{L} << 2^{L^2} = N_{total}$$

Minimal Number of Basis States Needed

$$|\psi\rangle = \sum_{k=1}^{N \ll N_{total}} a_k |k\rangle$$

basis states

Area Law of Entanglement Entropy



Entanglement between A and B

$$S \propto L \ \propto \ln N$$

$$N \sim 2^{L} \ll 2^{L^2} = N_{total}$$

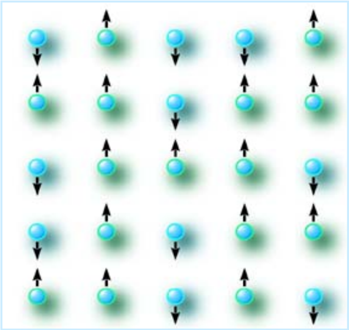
↑

Minimal Number of Basis States Needed

Is there any wave function automatically satisfying this area law

The Answer: Tensor Network State (Tensor Product State)

Verstraete, Cirac, arXiv:0407066



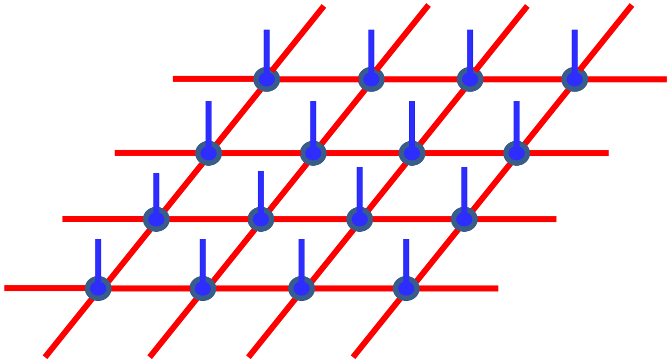
$$|\Psi\rangle = Tr \prod T_{x_i x'_i y_i y'_i}[m_i] |m_i\rangle$$

Local tensor

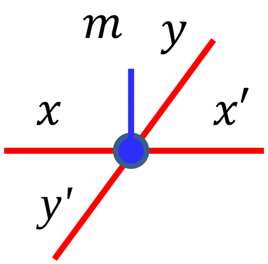
Virtual basis

Physical basis

Dimension D

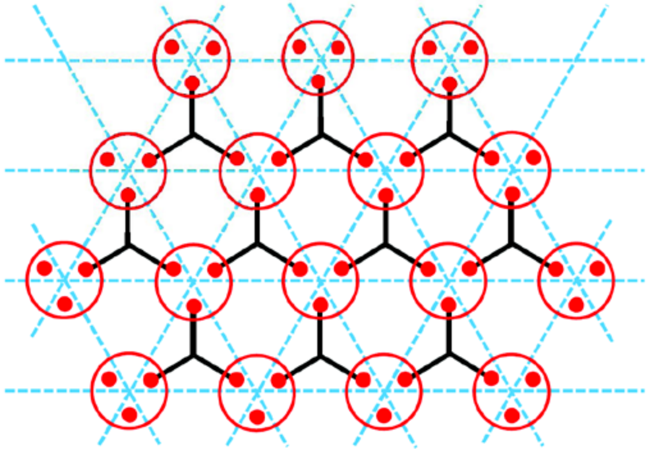


$$T_{xx'yy'}[m] =$$



PESS on other lattices

Triangular Lattice



Order of local tensors:

Simplex tensor: D^3

Projection tensor: dD^3

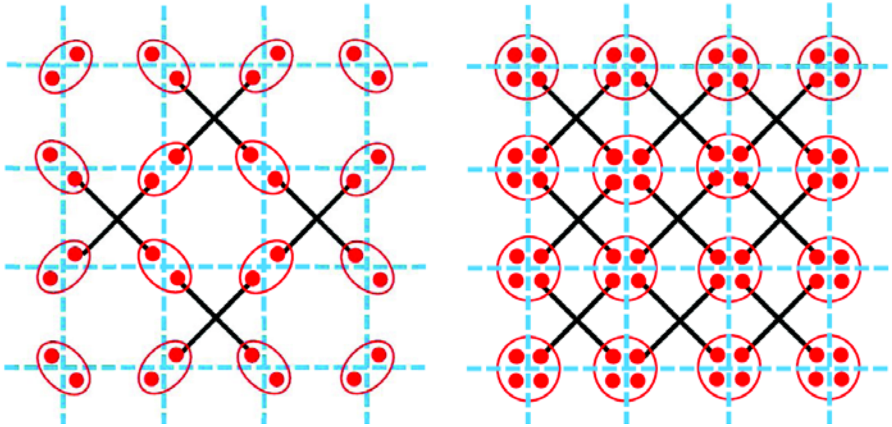
Order of local tensor in PEPS: dD^6

Square Lattice

Two kinds of simplex solid states

Vertex-sharing

Edge-sharing



References

- **Entanglement mean field theory**

  H.C. Jiang, et al, PRL **101**, 090603 (2008)

- **Second renormalization**

  Z. Y. Xie et al, PRL **103**, 160601 (2009)

- **Tensor representation of statistical models**

  H. H. Zhao, et al, PRB **81**, 174411 (2010)

- **Tensor renormalization using higher-order singular value decomposition**

  Z. Y. Xie et al, PRB **86**, 045139 (2012)

- **Projected entangled simplex states**

  Z. Y. Xie et al, PRX **4**, 011025 (2014)

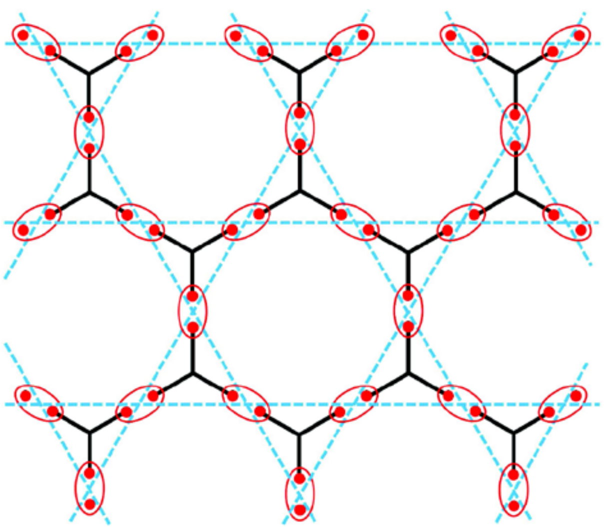
S=2 Simplex Solid State on the Kagome Lattice

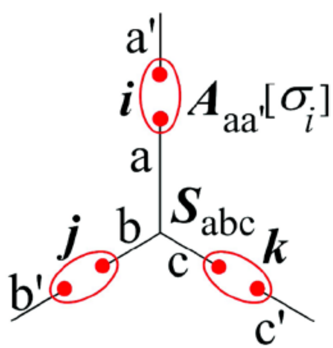
Local tensors

$$|0,0\rangle = \frac{1}{\sqrt{6}} \sum_{s_i s_j s_k} \varepsilon_{s_i s_j s_k} |s_i\rangle |s_j\rangle |s_k\rangle$$

$$S_{ijk} = \varepsilon_{ijk}$$ antisymmetric tensor

$$A_{ab}[\sigma] = \begin{pmatrix} 1 & 1 & 2 \\ a & b & \sigma \end{pmatrix}$$ C-G coefficients



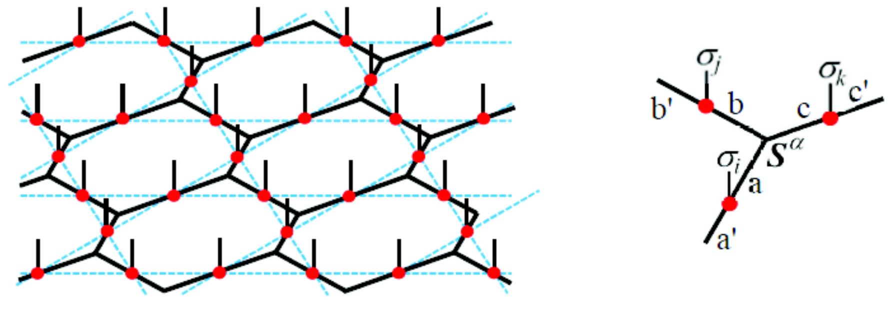


Projection tensor

Simplex tensor

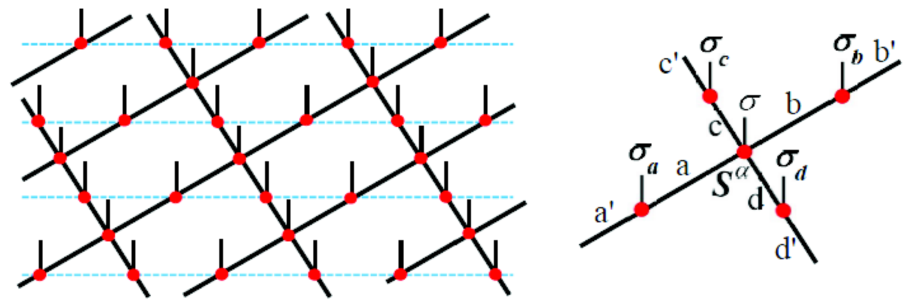
Projected Entangled Simplex State (PESS) on the Kagome lattice

3-PESS: a decorated honeycomb lattice



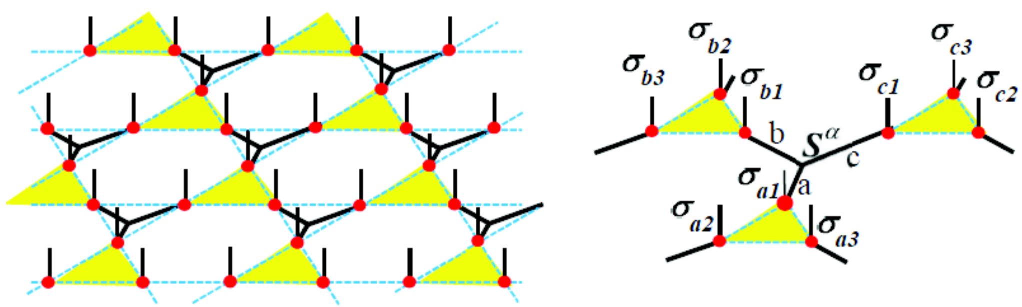
(a) 3-PESS

5-PESS: a decorated square lattice



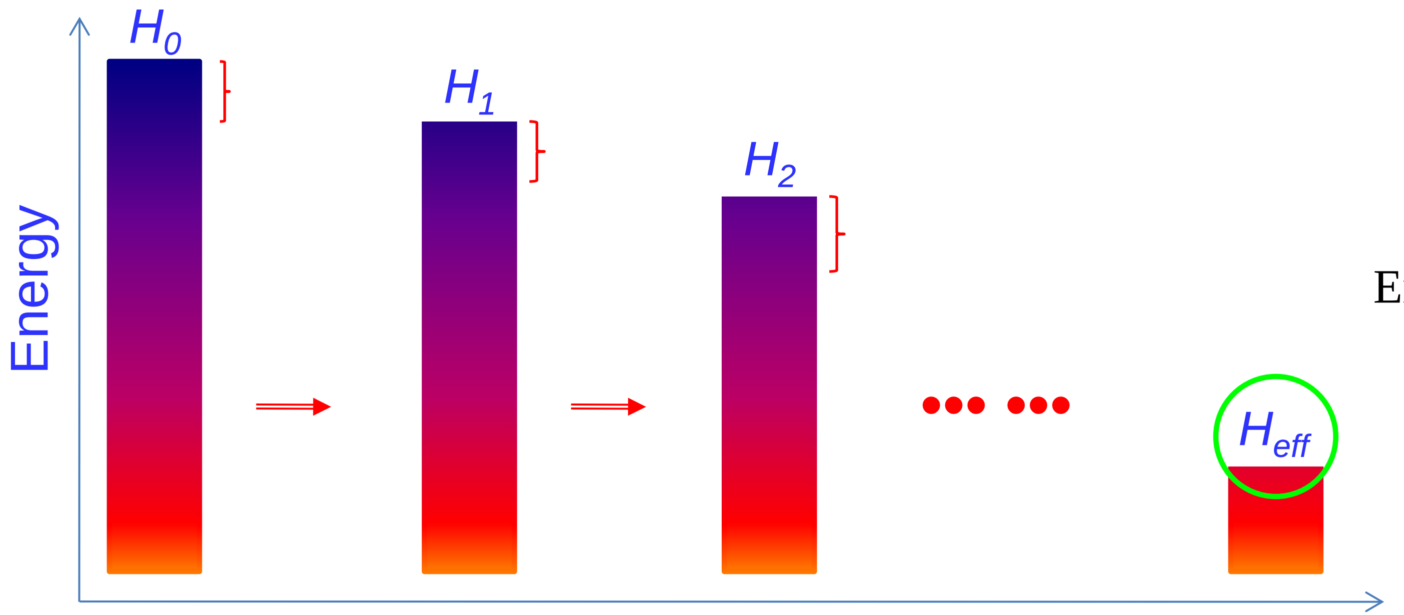
(b) 5-PESS

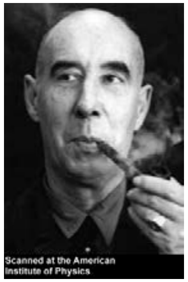
9-PESS: a honeycomb lattice



(c) 9-PESS

What Is Renormalization Group?





Ernst Stueckelberg

Physics: compression of basis space (phase space)

Mathematics: low rank approximation of matrix or tensor