Simulating 2d systems with PEPS

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Outline

1) Entanglement and 1d MPS

2) 2d PEPS

2.1) Basics

2.2) Computing

2.3) Fermions

3) Some further topics

Outline

1) Entanglement and 1d MPS

2) 2d PEPS

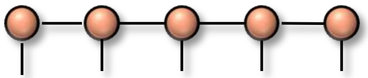
2.1) Basics

2.2) Computing

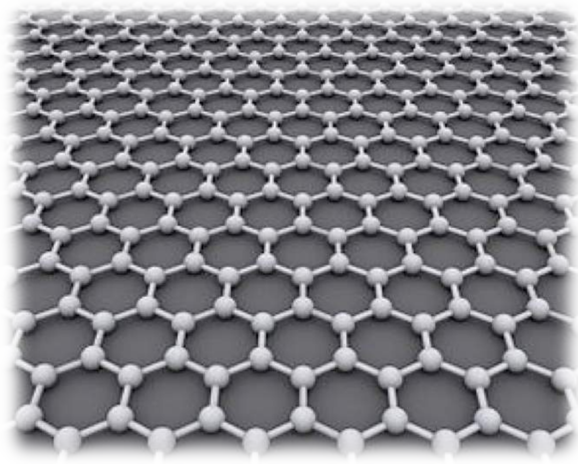
2.3) Fermions

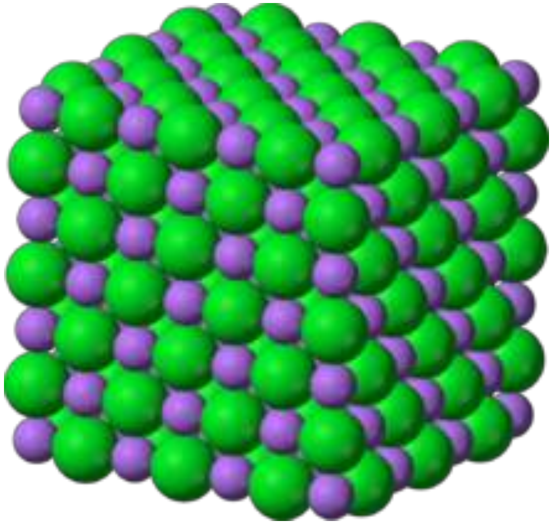
3) Some further topics

From MPS to PEPS

Matrix Product States (MPS)  *1d systems*

But we want to go beyond 1d systems!!!

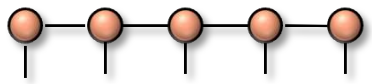




Very painful for DMRG...

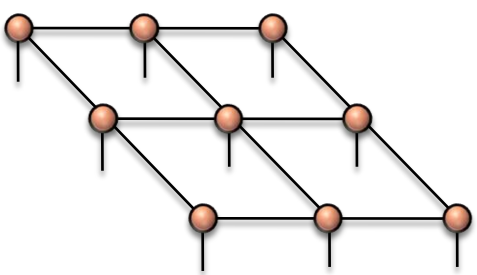
From MPS to PEPS

Matrix Product States (MPS)

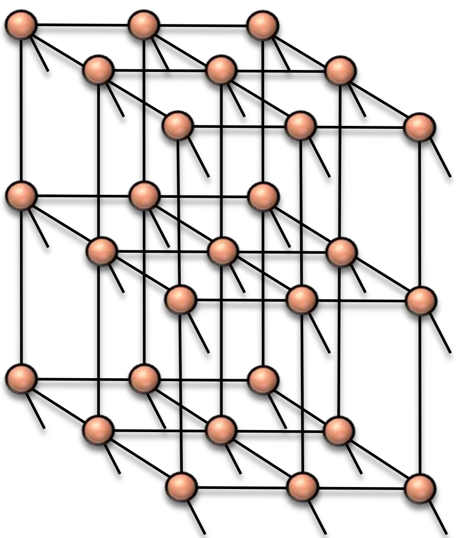


1d systems

Projected Entangled Pair States (PEPS), Tensor Product States (TPS)



2d systems

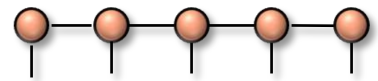


3d systems

and so on…

From MPS to PEPS

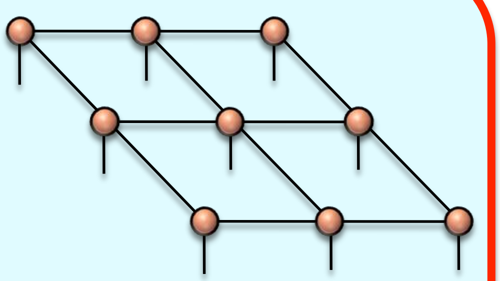
Matrix Product States (MPS)



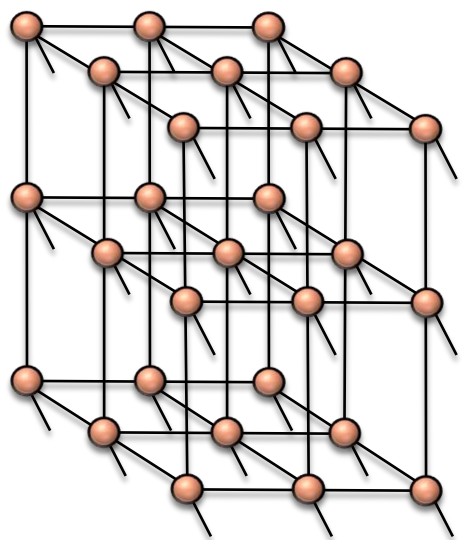
1d systems

Projected Entangled Pair States (PEPS),
Tensor Product States (TPS)

This lecture



2d systems



3d systems

and so on…

c.f. P. Corboz's talk

PEPS are not your friends... *(M. Lubasch)*

MPS

PEPS are not your friends… *(M. Lubasch)*

MPS

PEPS *(initially)*

PEPS are not your friends... *(M. Lubasch)*

MPS

PEPS *(initially)*

PEPS *(in the end)*

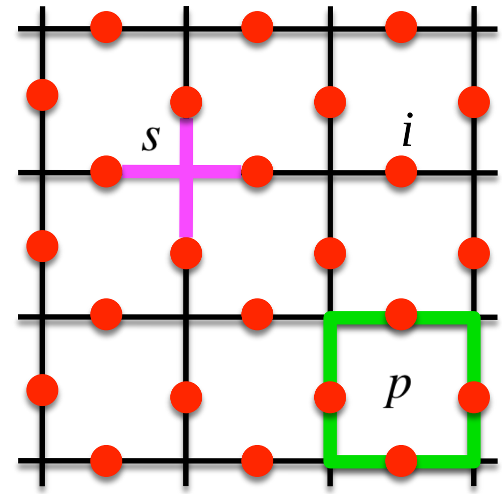
...but, after a lot of gymnastics,
they can be your allies!

Two exact examples

An exact example: Kitaev's Toric Code

Kitaev, 1997

$$H = -J\sum_s A_s - J\sum_p B_p$$

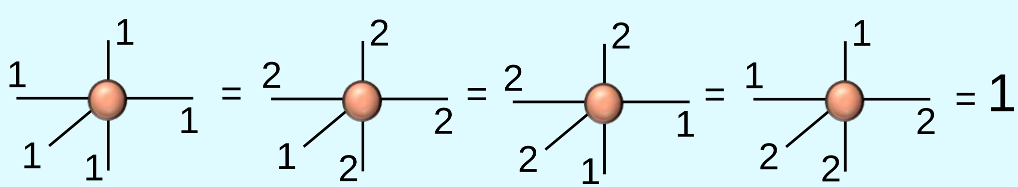


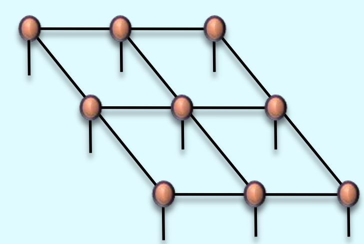
$$A_s = \prod_{i\in s}\sigma_i^x$$ *star operator*

$$B_p = \prod_{i\in p}\sigma_i^z$$ *plaquette operator*

Simplest known model with "topological order"

Ground state (and in fact all eigenstates) are PEPS with D=2

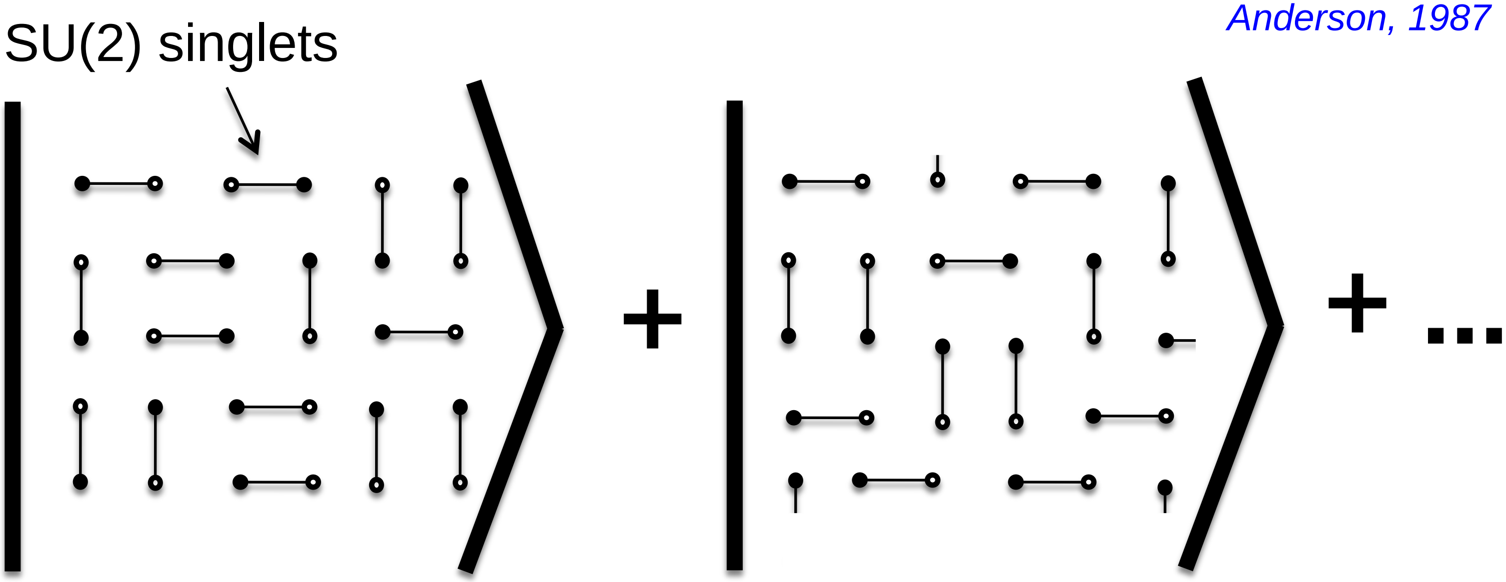




And another tensor rotated 90˚

Resonating Valence Bond State

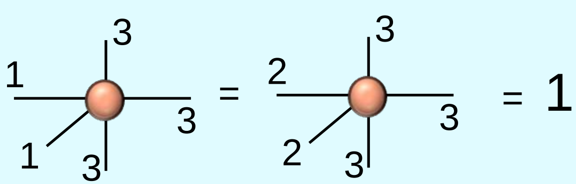
Anderson, 1987



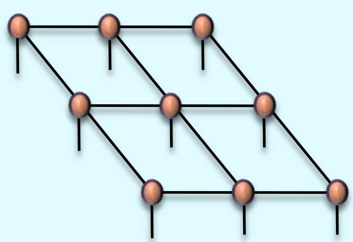
Equal superposition of all possible nearest-neighbor singlet coverings of a lattice (spin liquid)

Proposed to understand high-T_C superconductivity

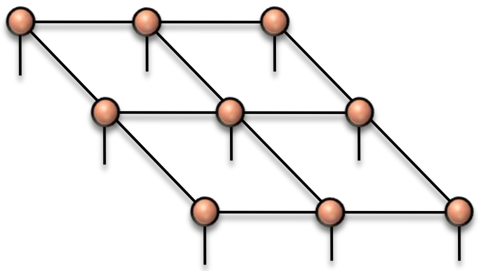
It is a PEPS with D=3



And rotations

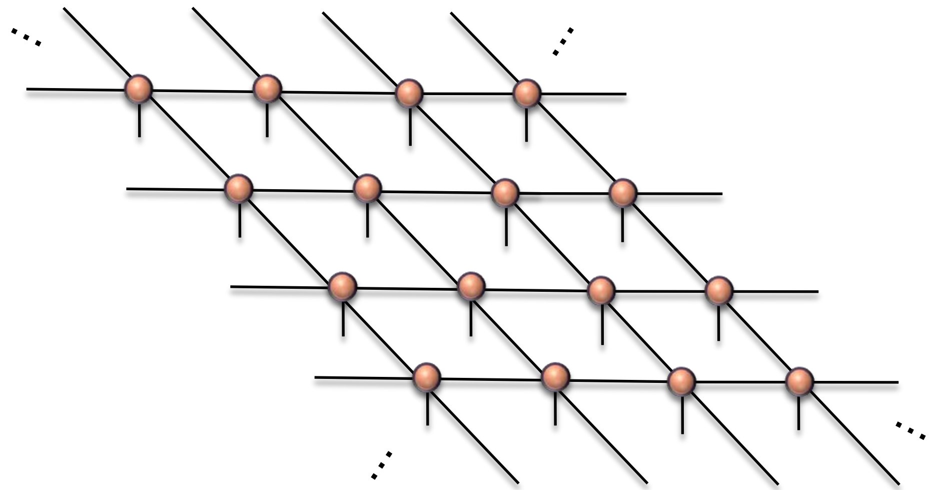


PEPS...



F. Verstraete, I. Cirac, cond-mat/0407066

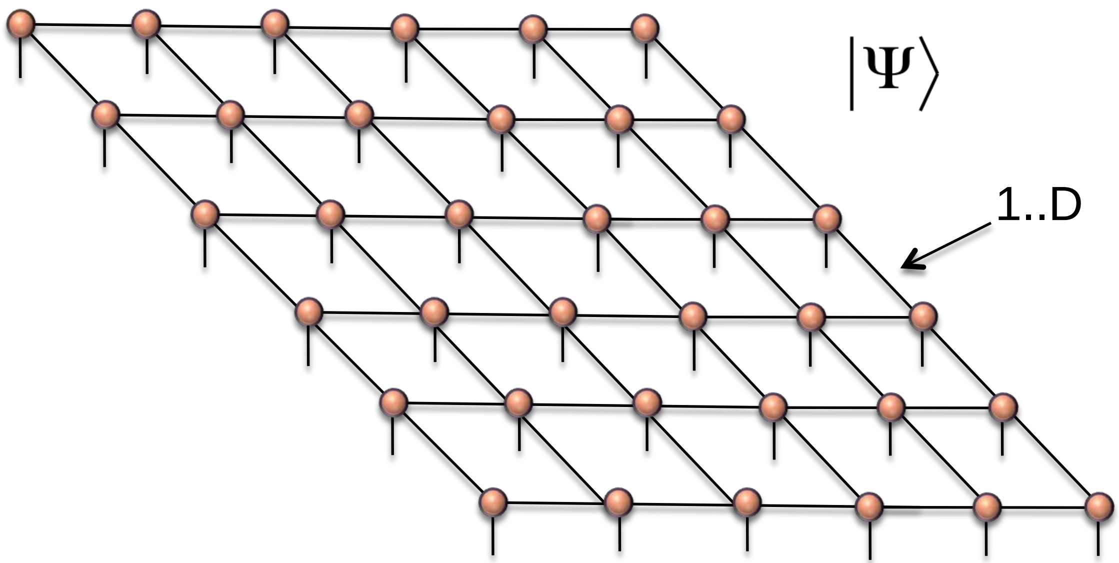
and infinite PEPS (iPEPS)

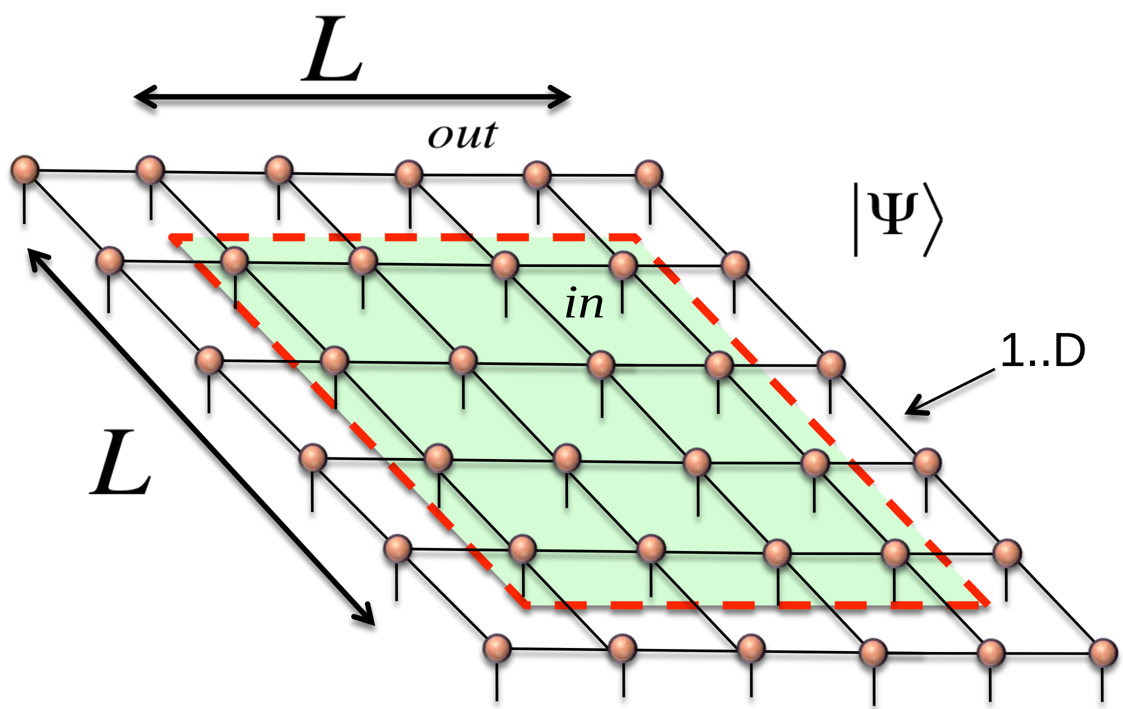


assuming translation invariance

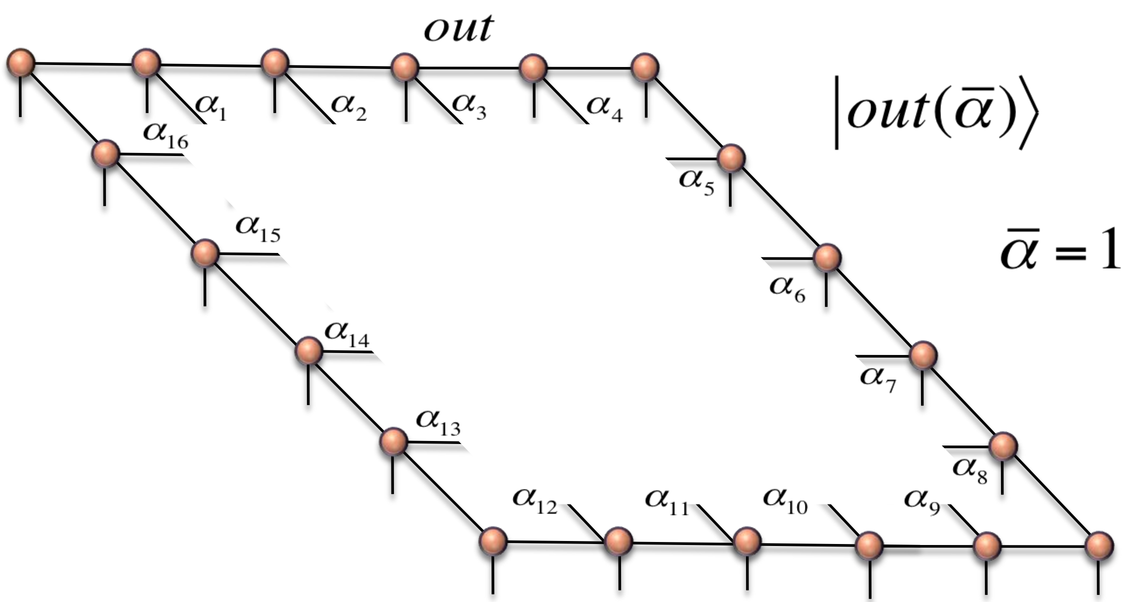
J. Jordan, RO, G. Vidal, F. Verstraete, I. Cirac, Phys. Rev. Lett. 101, 250602 (2008)

PEPS obey 2d area-law


$|\Psi\rangle$
1..D

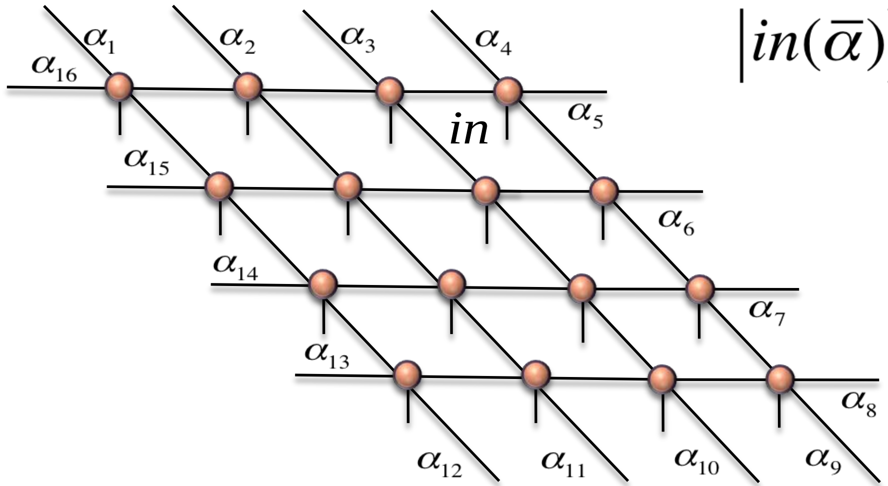


L
out
$|\Psi\rangle$
in
1..D
L



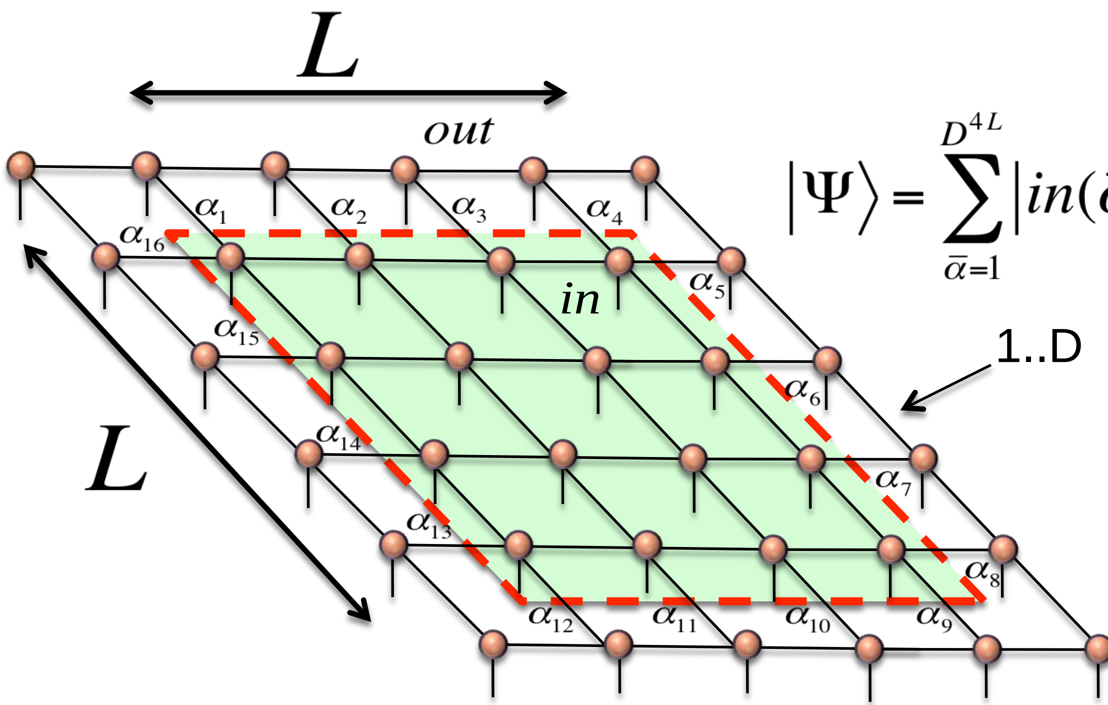
$$|out(\bar{\alpha})\rangle$$

$$\bar{\alpha} = 1, 2, \ldots, D^{4L}$$



$$|in(\bar{\alpha})\rangle$$

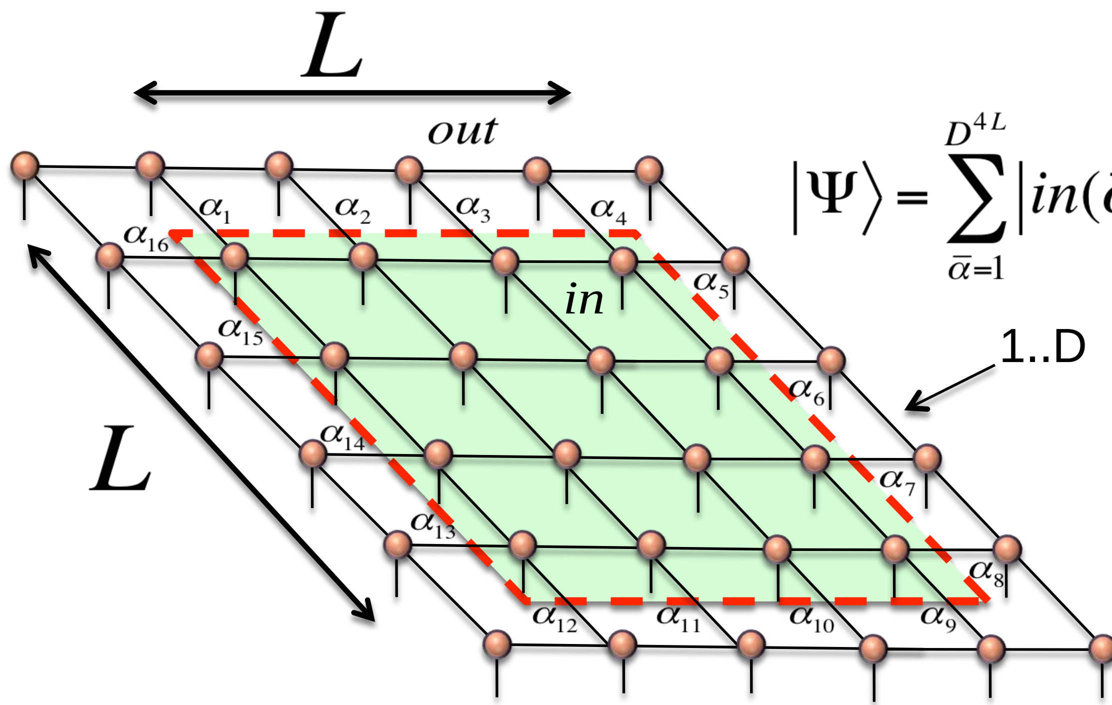
$$\bar{\alpha} = 1, 2, \ldots, D^{4L}$$



$$|\Psi\rangle = \sum_{\bar{\alpha}=1}^{D^{4L}} |in(\bar{\alpha})\rangle |out(\bar{\alpha})\rangle$$

$$\rho_{in} = \mathrm{tr}_{out}\left(|\Psi\rangle\langle\Psi|\right) = \sum_{\bar{\alpha},\bar{\alpha}'} X_{\bar{\alpha},\bar{\alpha}'} |in(\bar{\alpha})\rangle\langle in(\bar{\alpha}')| \qquad X_{\bar{\alpha},\bar{\alpha}'} = \langle out(\bar{\alpha}')|out(\bar{\alpha})\rangle$$

$$rank(\rho_{in}) \leq D^{4L} \qquad S(L) = -\mathrm{tr}\left(\rho_{in} \log \rho_{in}\right) \leq \log(D)4L$$



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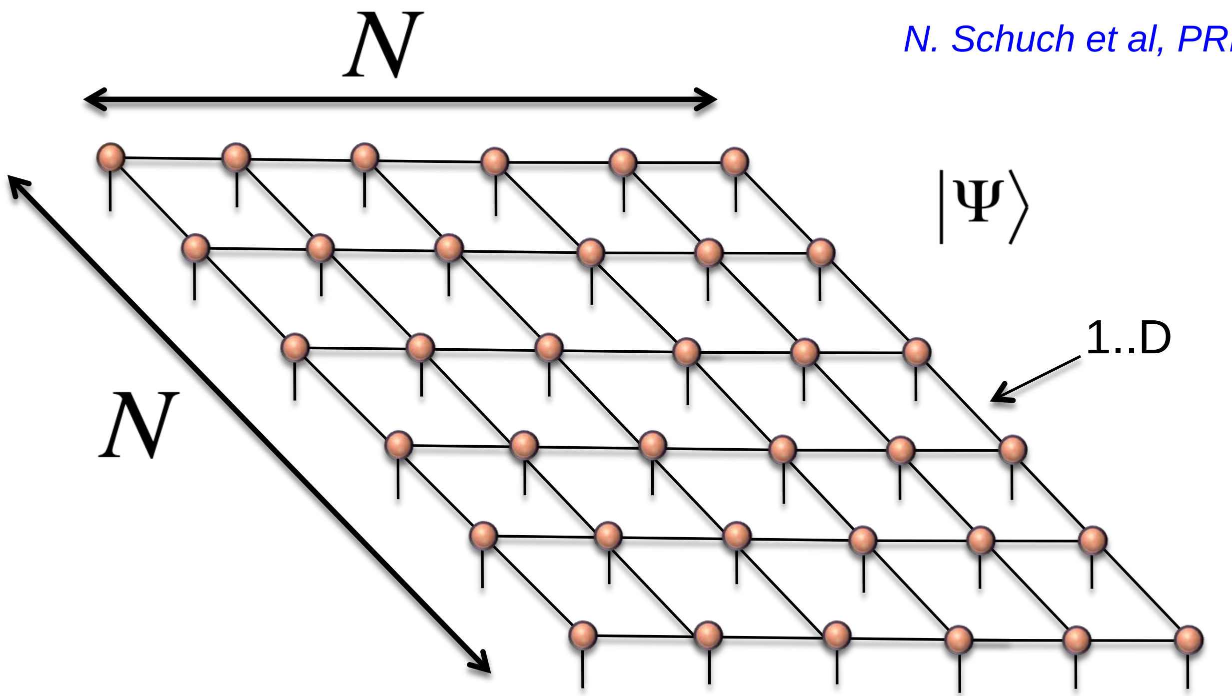
$$rank(\rho_{in}) \le D^{4L} \qquad S(L) = -\mathrm{tr}\left(\rho_{in}\log\rho_{in}\right) \le \log(D)\boxed{4L}$$

prefactor

size of the boundary

Exact contraction is inneficient

N. Schuch et al, PRL 98, 140506 (2007)



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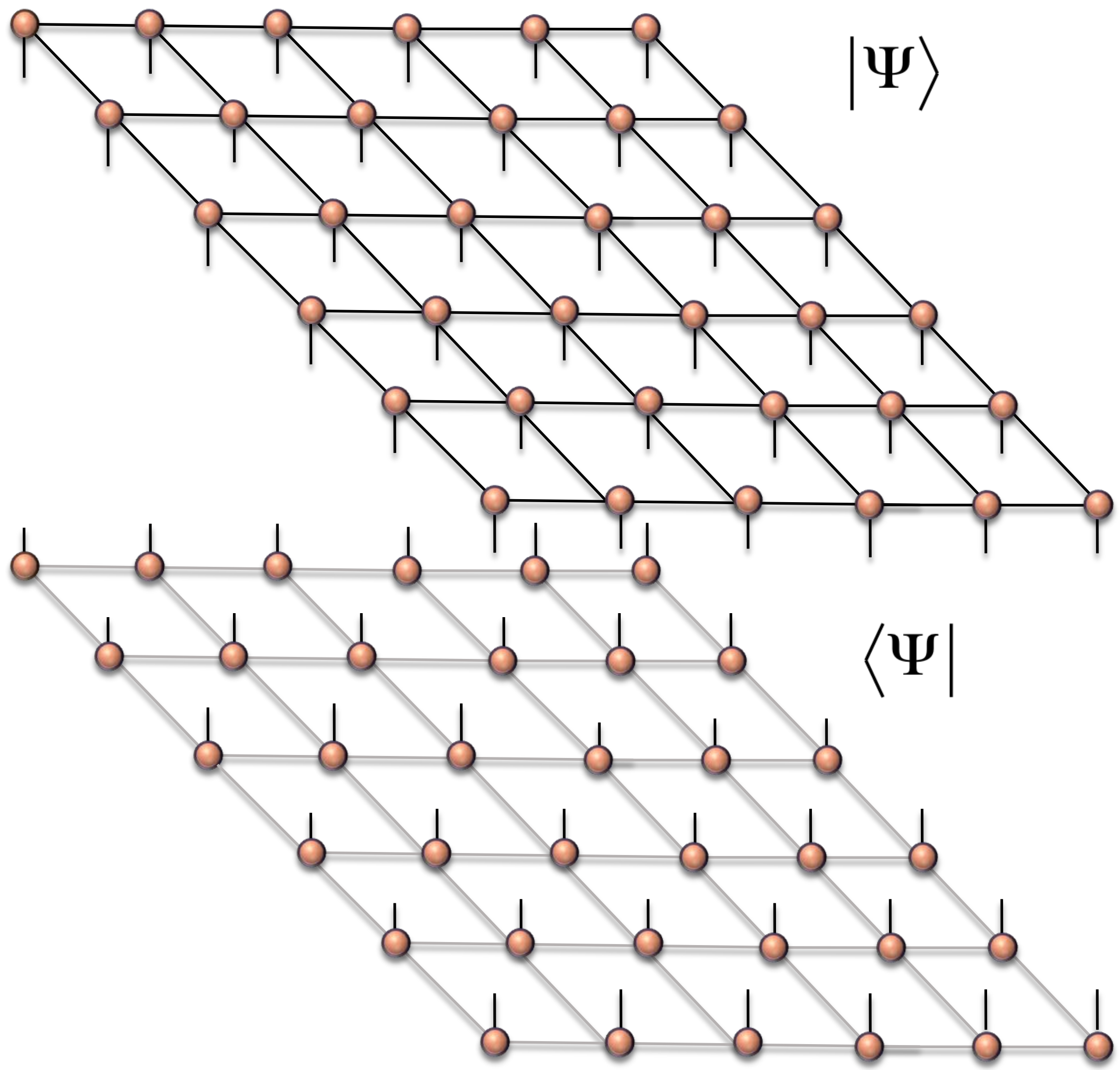

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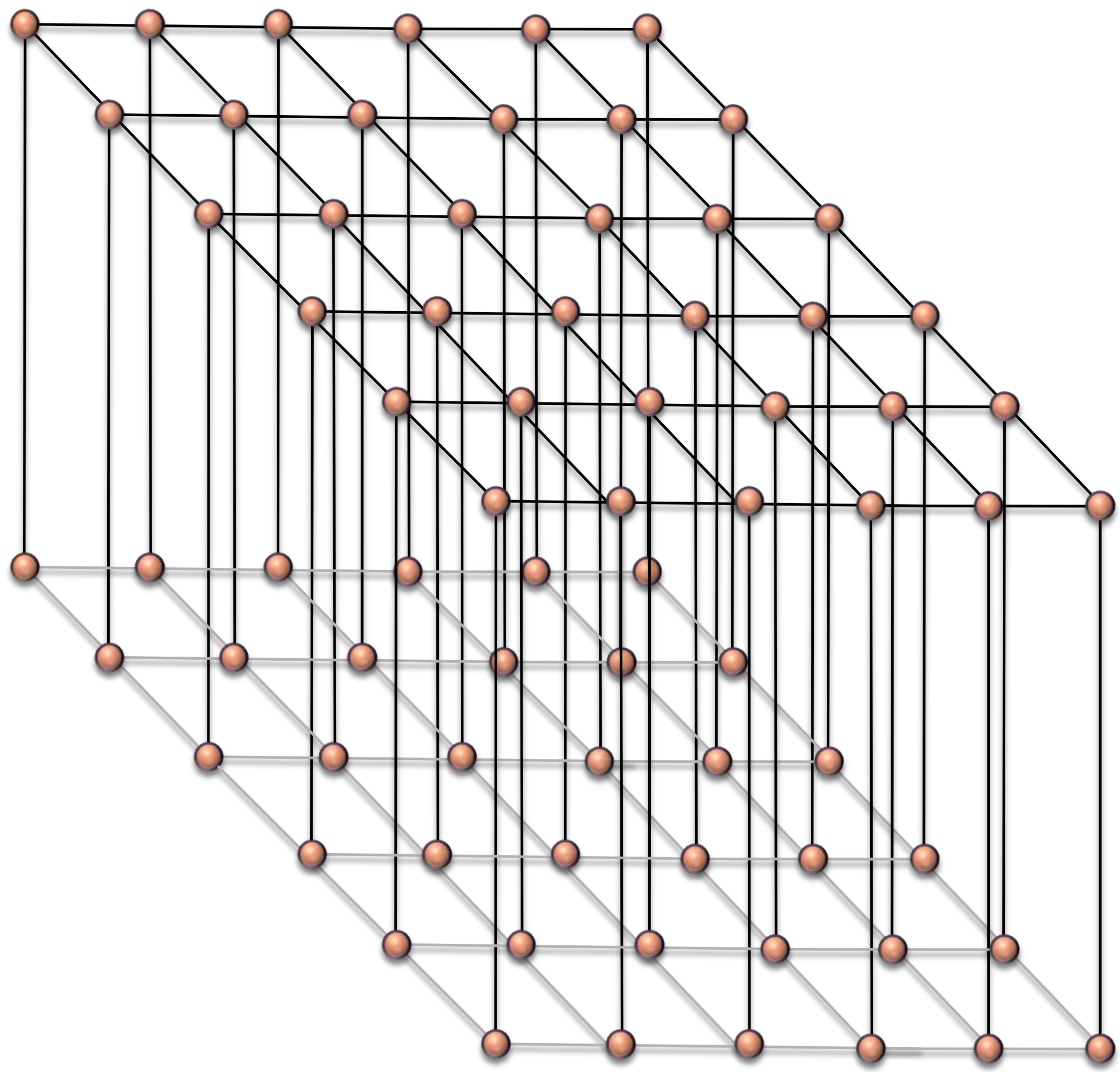
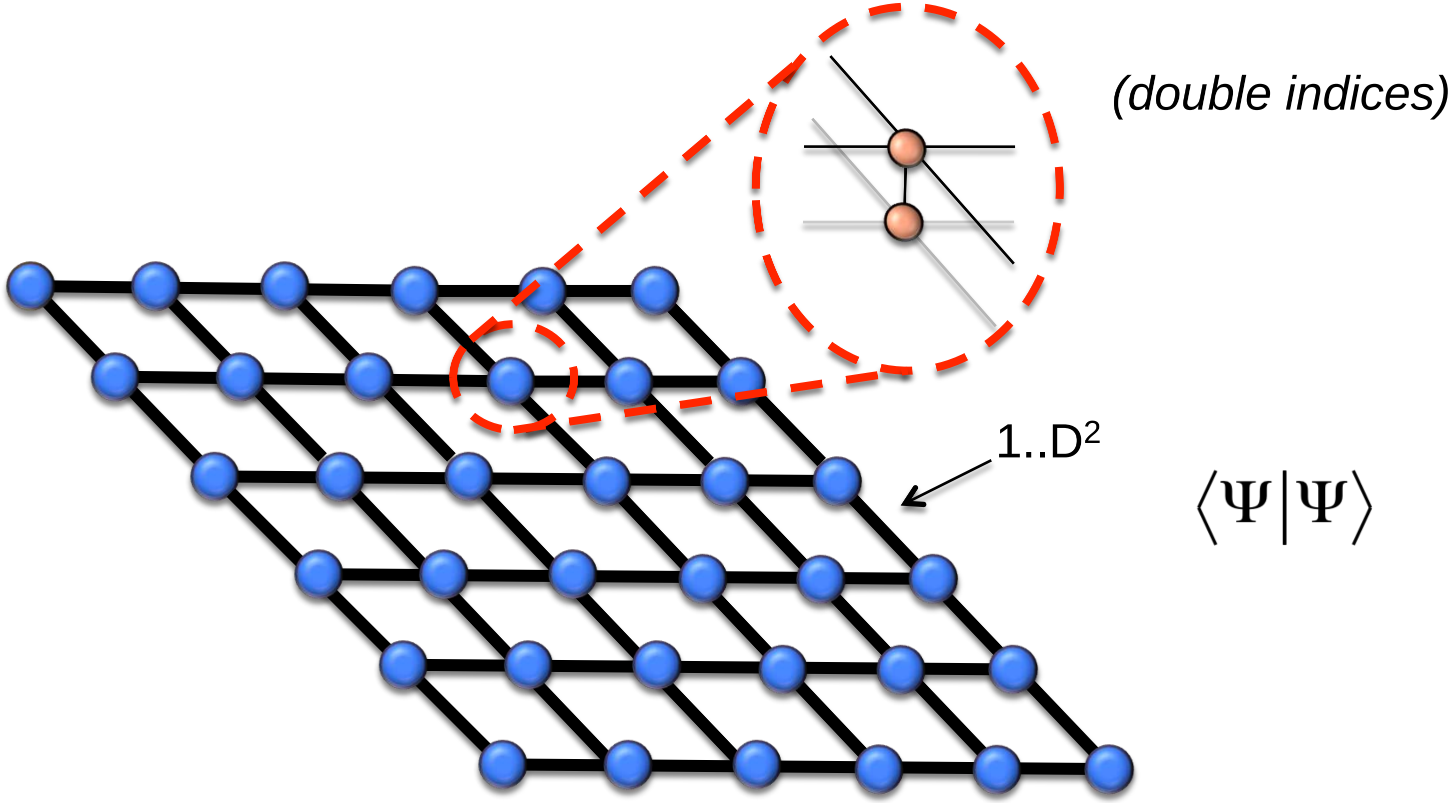


$$\langle \Psi | \Psi \rangle$$

Exact contraction is ineficient

N. Schuch et al, PRL 98, 140506 (2007)



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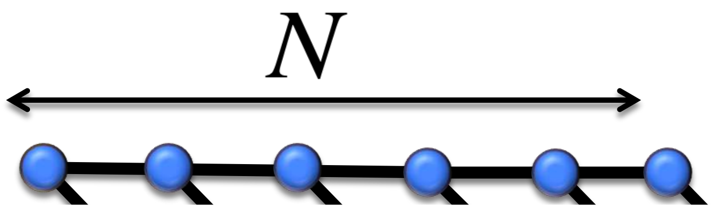

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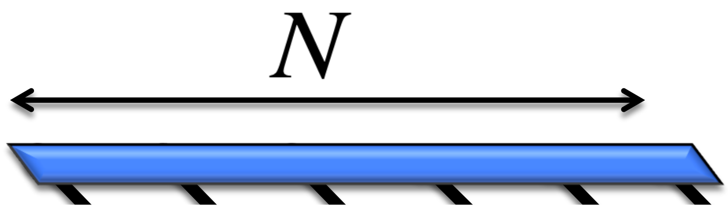
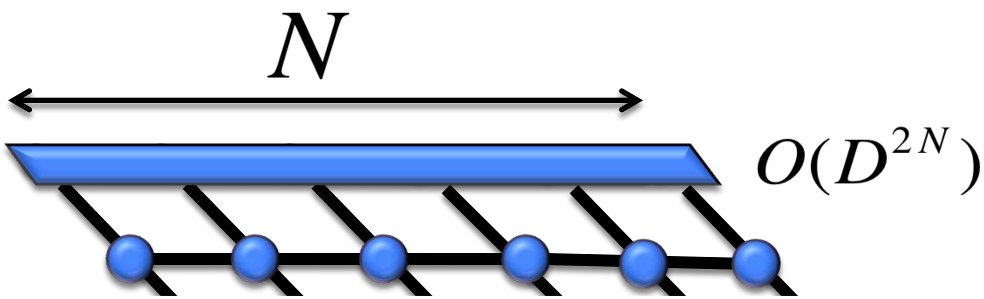


$O(D^{2N})$

Exact contraction is ineficient

N. Schuch et al, PRL 98, 140506 (2007)



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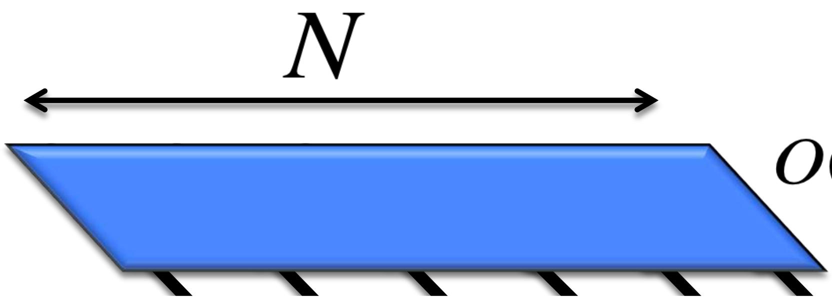
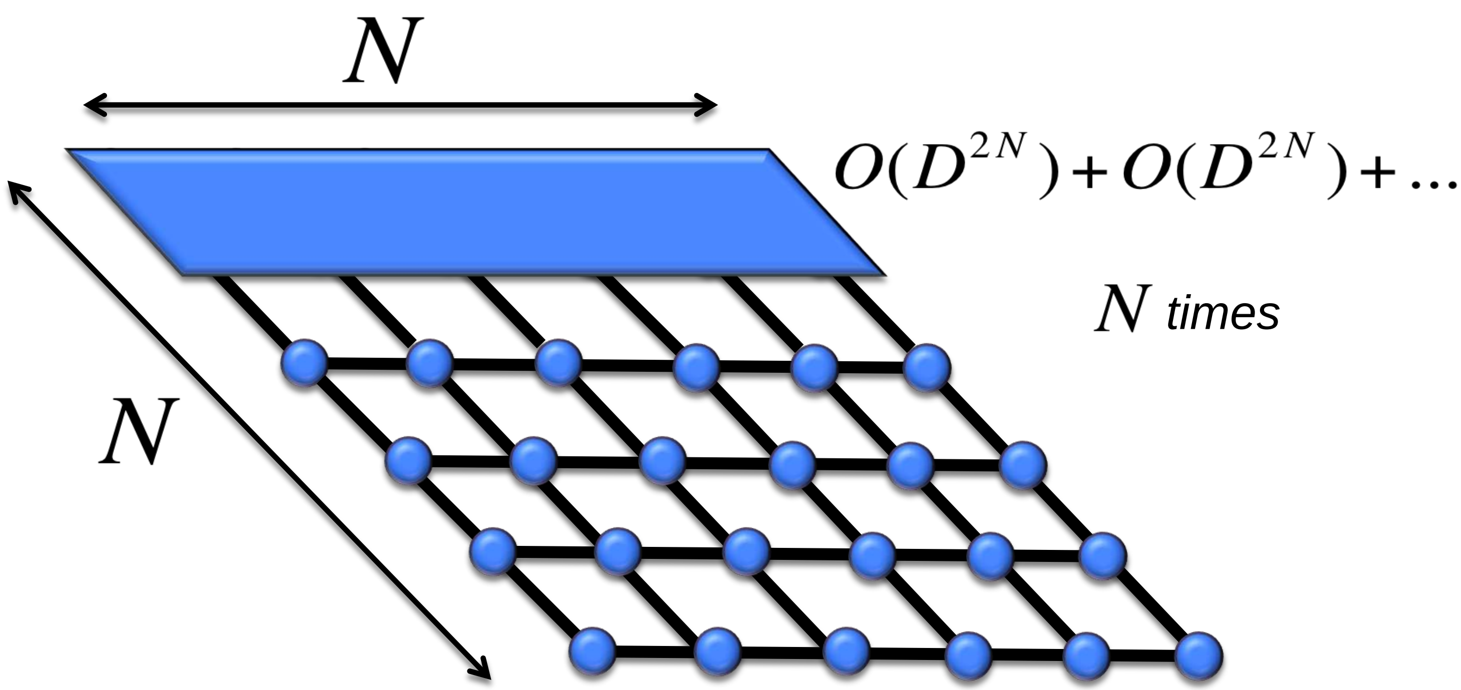


$$O(D^{2N}) + O(D^{2N})$$

Exact contraction is ineficient

N. Schuch et al, PRL 98, 140506 (2007)



computing time ~ $O(ND^{2N})$ **Exponential amount of time!**

Mathematical statement: *exact contraction of a PEPS is a #P-Hard problem (harder than NP-Complete)*

Applies also to expectation values of observables

Critical correlation functions

F. Verstraete et al, PRL 96, 220601 (2006)

$$\left|\Psi(\beta)\right\rangle = \frac{1}{\sqrt{Z(\beta)}}\exp\left(\frac{\beta}{2}\sum_{\langle i,j\rangle}\sigma_z^i\sigma_z^j\right)\left|+,+\ldots+\right\rangle$$

Expectation values are those of the classical 2d Ising model

$$\left\langle\sigma_z^r\sigma_z^{r'}\right\rangle_\beta = \frac{1}{Z(\beta)}\sum_{\{s\}}s^r s^{r'}\exp\left(\beta\sum_{\langle i,j\rangle}s^i s^j\right) \qquad s=\pm 1$$

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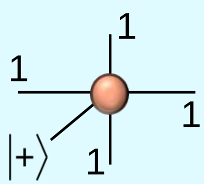
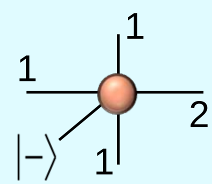
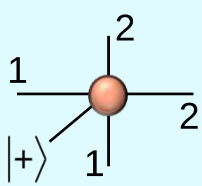
It is a PEPS with D=2 *(left as exercise):*



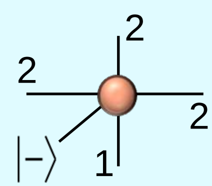
$$= \left(\cosh(\beta/2)\right)^4$$



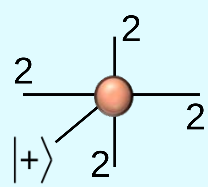
$$= \left(\cosh(\beta/2)\right)^3\left(\sinh(\beta/2)\right)$$



$$= \left(\cosh(\beta/2)\right)^2\left(\sinh(\beta/2)\right)^2$$



$$= \left(\cosh(\beta/2)\right)\left(\sinh(\beta/2)\right)^3$$



$$= \left(\sinh(\beta/2)\right)^4$$

+ permutations

At $\beta_c = \left(\log(1+\sqrt{2})\right)/2$ ***the correlation length is infinite:*** $\left\langle \sigma_z^r\sigma_z^{r'}\right\rangle_{\beta_c} \approx \dfrac{a}{\left|r-r'\right|^{1/4}}$

PEPS to/from Hamiltonians

PEPS to/from Hamiltonians

World of Hamiltonians

World of states

PEPS to/from Hamiltonians

World of Hamiltonians

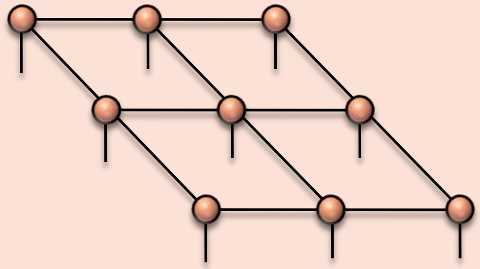
Ground state of a gapped Parent Hamiltonian with local interactions

$$H_{parent} = \sum h_{local}$$

e.g. D. Perez-Garcia et al, QIC 8, 0650 (2008)

World of states

Generic PEPS with finite D



PEPS to/from Hamiltonians

World of Hamiltonians

Ground state of a gapped Parent Hamiltonian with local interactions

$$H_{parent} = \sum h_{local}$$

Hamiltonian with local interactions

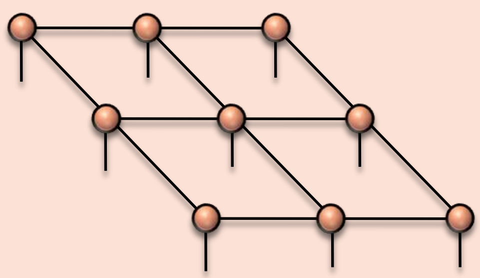
$$H' = \sum h'_{local}$$

e.g. D. Perez-Garcia et al, QIC 8, 0650 (2008)

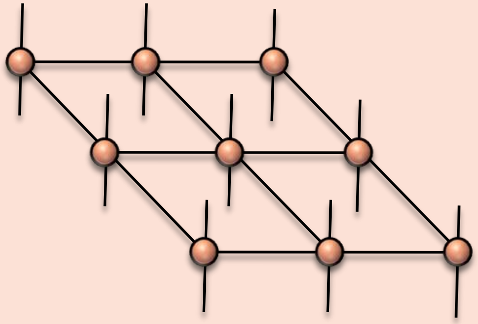
M. Hastings, PRB 73, 085115 (2006)

World of states

Generic PEPS with finite D

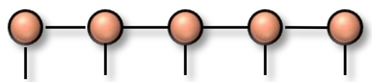
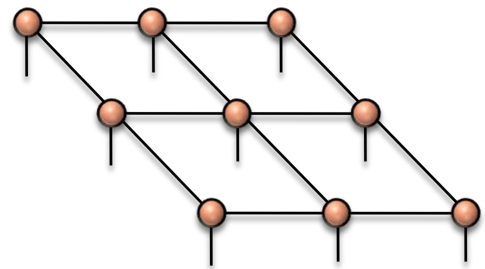
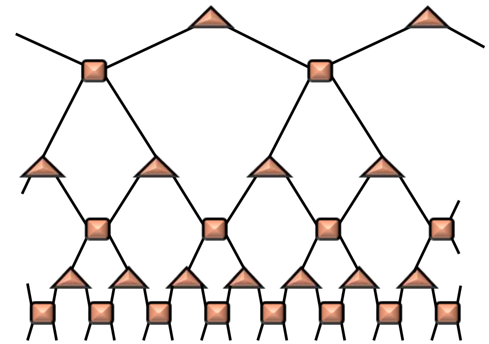


Thermal states can be approx. by a PEPS with finite D



PEPS target the relevant corner of the Hilbert space (area-law)

Comparison

	MPS in 1d 	*PEPS in 2d* 	*MERA in 1d* 
Ent. entropy	$S(L) = O(1)$	$S(L) = O(L)$	$S(L) = O(\log L)$
Exact contraction	efficient	inefficient	efficient
Corr. length	finite	finite & infinite	finite & infinite
To/from	1d Ham.	2d Ham.	1d Ham.
Tensors	arbitrary	arbitrary	constrained

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PEPS as ansatz:
variational optimization

Variational optimization (e.g. finite PEPS)

e.g. F. Verstraete, I. Cirac, cond-mat/0407066

$$\min\left(\frac{\langle\Psi|H|\Psi\rangle}{\langle\Psi|\Psi\rangle}\right)$$

Optimize over each tensor individually and sweep over the entire system (as in DMRG)

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e.g. F. Verstraete, I. Cirac, cond-mat/0407066

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Optimize over each tensor individually and sweep over the entire system (as in DMRG)

$|\Psi\rangle$

E

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e.g. F. Verstraete, I. Cirac, cond-mat/0407066

$$\min\left(\frac{\langle\Psi|H|\Psi\rangle}{\langle\Psi|\Psi\rangle}\right)$$

Optimize over each tensor individually and sweep over the entire system (as in DMRG)

$$|\Psi\rangle \xrightarrow[A^1]{\min} |\Psi^1\rangle$$

$$E \quad \geq \quad E^1$$

Variational optimization (e.g. finite PEPS)

e.g. F. Verstraete, I. Cirac, cond-mat/0407066

$$\min\left(\frac{\langle\Psi|H|\Psi\rangle}{\langle\Psi|\Psi\rangle}\right)$$

Optimize over each tensor individually and sweep over the entire system (as in DMRG)

$$|\Psi\rangle \xrightarrow{\min_{A^1}} |\Psi^1\rangle \xrightarrow{\min_{A^2}} \ldots \xrightarrow{\min_{A^N}} |\Psi^N\rangle$$

$$E \geq E^1 \geq \ldots \geq E^N$$

Variational optimization (e.g. finite PEPS)

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$$\min\left(\frac{\langle\Psi|H|\Psi\rangle}{\langle\Psi|\Psi\rangle}\right)$$

Optimize over each tensor individually and sweep over the entire system (as in DMRG)

$$|\Psi\rangle \xrightarrow{\min_{A^1}} |\Psi^1\rangle \xrightarrow{\min_{A^2}} \cdots \xrightarrow{\min_{A^N}} |\Psi^N\rangle \xrightarrow{\min_{A^{N-1}}} |\Psi^{N+1}\rangle$$

$$E \geq E^1 \geq \cdots \geq E^N \geq E^{N+1}$$

Variational optimization (e.g. finite PEPS)

e.g. F. Verstraete, I. Cirac, cond-mat/0407066

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Optimize over each tensor individually and sweep over the entire system (as in DMRG)

$$|\Psi\rangle \xrightarrow{\min_{A^1}} |\Psi^1\rangle \xrightarrow{\min_{A^2}} \ldots \xrightarrow{\min_{A^N}} |\Psi^N\rangle \xrightarrow{\min_{A^{N-1}}} |\Psi^{N+1}\rangle \xrightarrow{\min_{A^{N-2}}} \ldots$$

$$E \geq E^1 \geq \ldots \geq E^N \geq E^{N+1} \geq \ldots$$

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$$\min\left(\frac{\langle\Psi|H|\Psi\rangle}{\langle\Psi|\Psi\rangle}\right)$$

Optimize over each tensor individually and sweep over the entire system (as in DMRG)

$$|\Psi\rangle \xrightarrow{\min_{A^1}} |\Psi^1\rangle \xrightarrow{\min_{A^2}} \cdots \xrightarrow{\min_{A^N}} |\Psi^N\rangle \xrightarrow{\min_{A^{N-1}}} |\Psi^{N+1}\rangle \xrightarrow{\min_{A^{N-2}}} \cdots$$

$$E \geq E^1 \geq \cdots \geq E^N \geq E^{N+1} \geq \cdots$$

$$A^{*i}$$

$$\frac{\partial}{\partial A^{*i}}\left(\langle\Psi|H|\Psi\rangle - \lambda\langle\Psi|\Psi\rangle\right) = 0$$

$$A^i$$

Minimization of quadratic function

Variational optimization (e.g. finite PEPS)

e.g. F. Verstraete, I. Cirac, cond-mat/0407066

$$\min\left(\frac{\langle\Psi|H|\Psi\rangle}{\langle\Psi|\Psi\rangle}\right)$$

Optimize over each tensor individually and sweep over the entire system (as in DMRG)

$$|\Psi\rangle \xrightarrow{\min_{A^1}} |\Psi^1\rangle \xrightarrow{\min_{A^2}} \ldots \xrightarrow{\min_{A^N}} |\Psi^N\rangle \xrightarrow{\min_{A^{N-1}}} |\Psi^{N+1}\rangle \xrightarrow{\min_{A^{N-2}}} \ldots$$

$$E \geq E^1 \geq \ldots \geq E^N \geq E^{N+1} \geq \ldots$$

$$A^{*i}$$

$$\frac{\partial}{\partial A^{*i}}\left(\langle\Psi|H|\Psi\rangle - \lambda\langle\Psi|\Psi\rangle\right) = 0$$

$$A^i$$

Minimization of quadratic function

$$\mathbf{H}^i_{eff}\vec{A}^i = \lambda\mathbf{N}^i\vec{A}^i$$

Generalized eigenvalue problem

Once $\mathbf{H}^i_{eff}$ and $\mathbf{N}^i$ are known, we can solve this problem efficiently

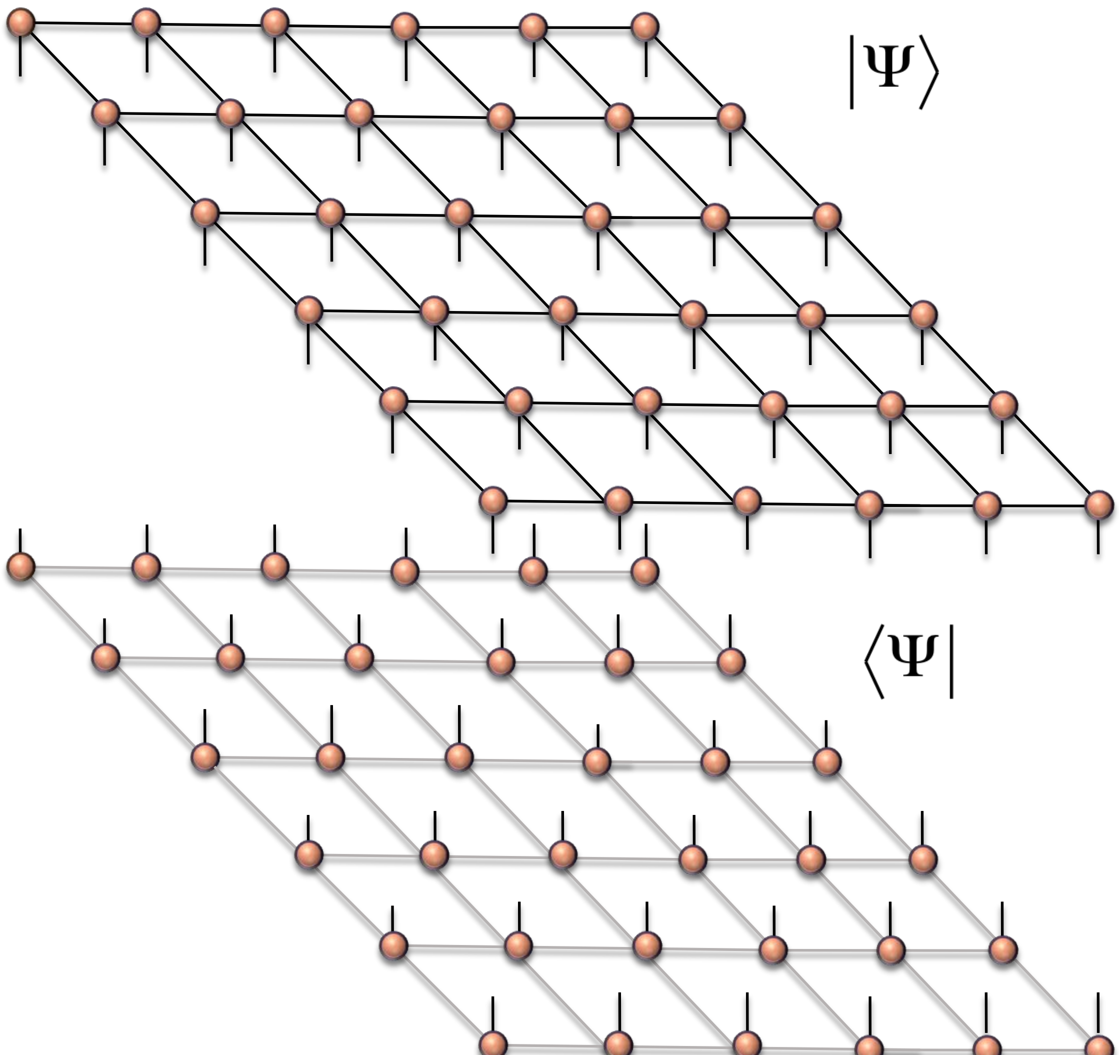
Approximate calculation of $\mathbf{H}^i_{eff}$ and $\mathbf{N}^i$

e.g. calculation of $\mathbf{N}^i \vec{A}^i$

$$\frac{\partial}{\partial A^{*i}} \left(\langle \Psi | H | \Psi \rangle - \lambda \langle \Psi | \Psi \rangle \right) = 0 \quad \Rightarrow \quad \mathbf{H}^i_{eff} \vec{A}^i = \lambda \mathbf{N}^i \vec{A}^i$$

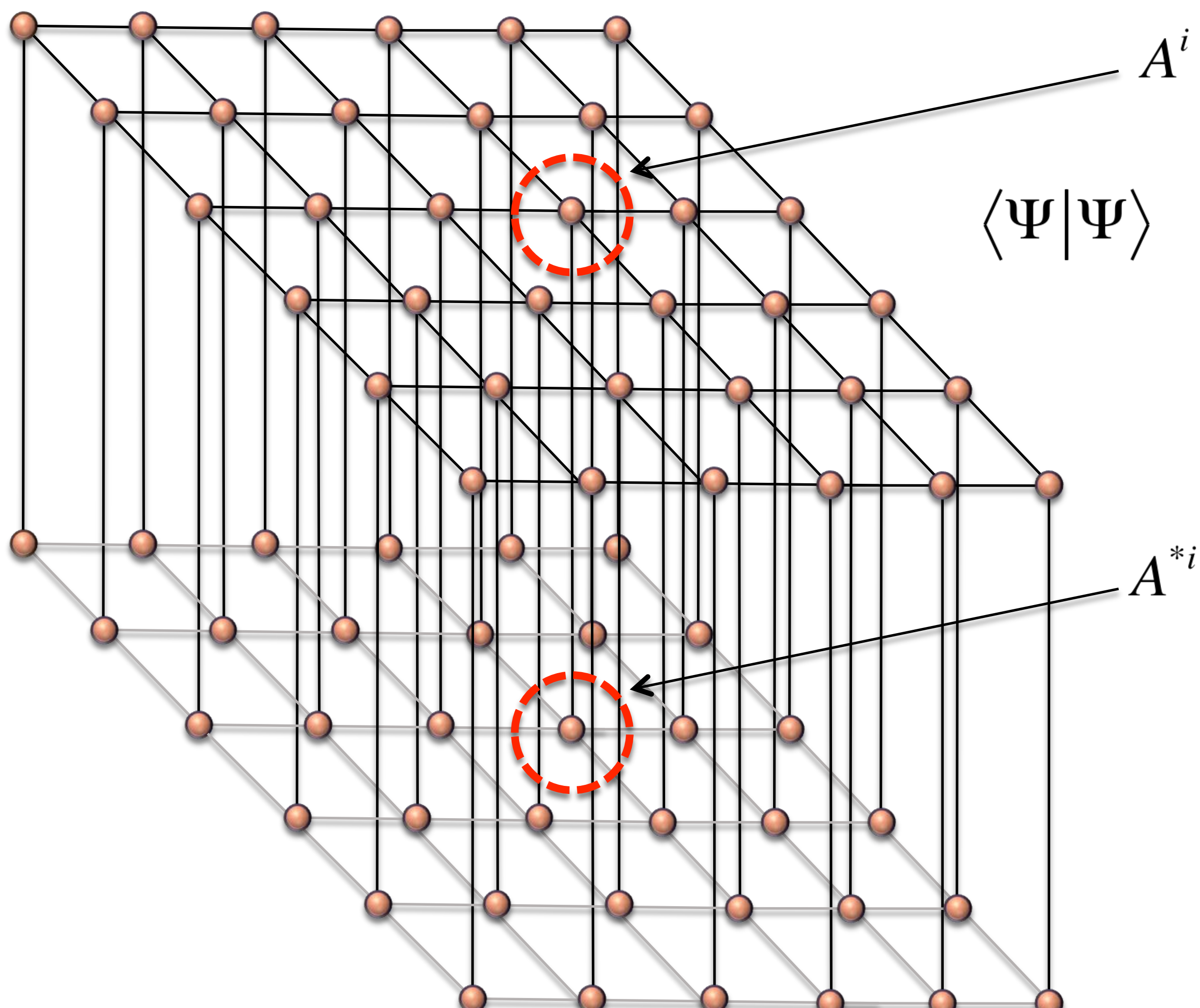
e.g. calculation of $\mathbf{N}^i \vec{A}^i$

$$\frac{\partial}{\partial A^{*i}}\left(\langle\Psi|H|\Psi\rangle - \lambda\langle\Psi|\Psi\rangle\right) = 0 \quad \Rightarrow \quad \mathbf{H}^i_{eff}\vec{A}^i = \lambda \mathbf{N}^i \vec{A}^i$$



e.g. calculation of $\mathbf{N}^i \vec{A}^i$

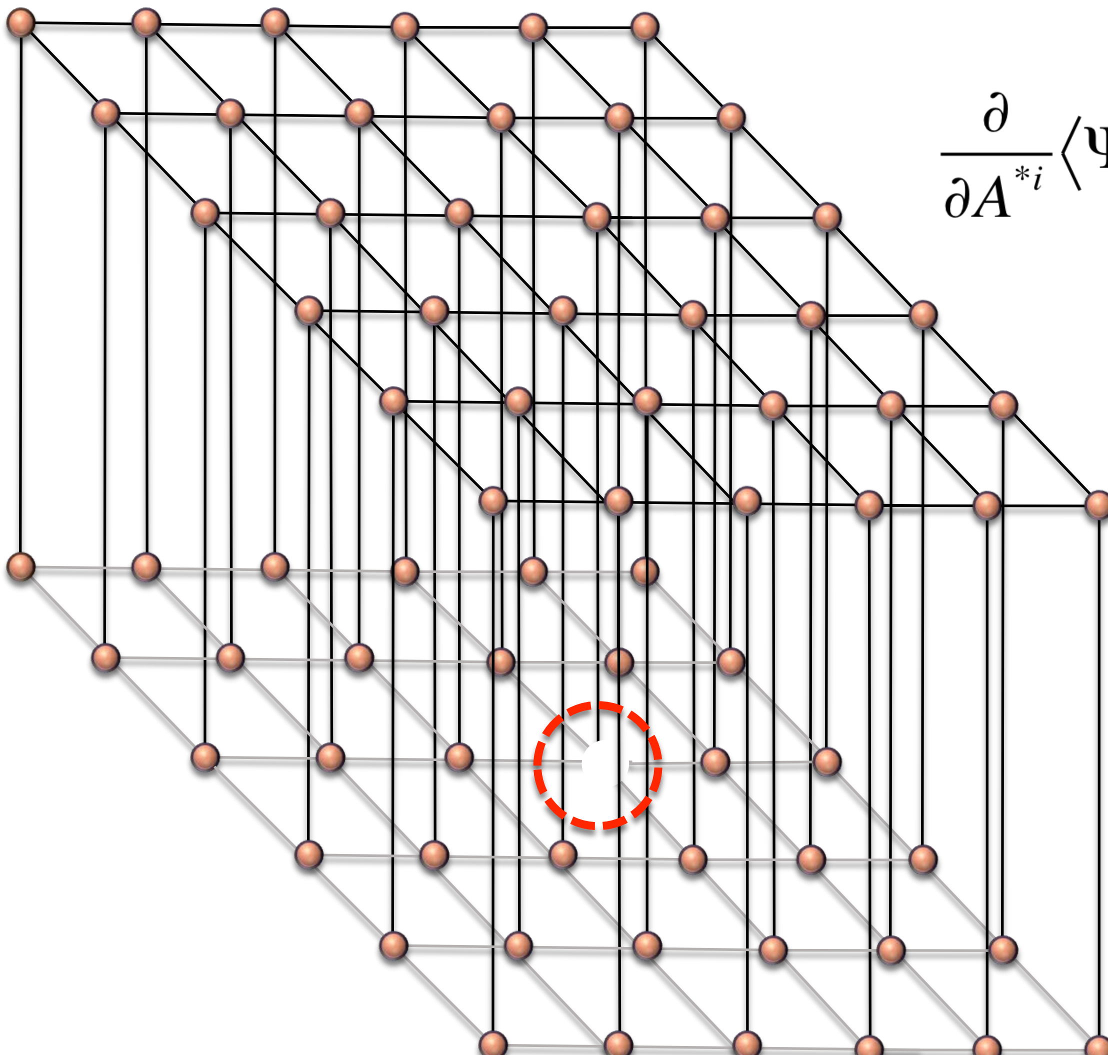
$$\frac{\partial}{\partial A^{*i}}\left(\langle\Psi|H|\Psi\rangle - \lambda\langle\Psi|\Psi\rangle\right) = 0 \quad \Rightarrow \quad \mathbf{H}^i_{eff}\vec{A}^i = \lambda \mathbf{N}^i \vec{A}^i$$



e.g. calculation of $\mathbf{N}^i \vec{A}^i$

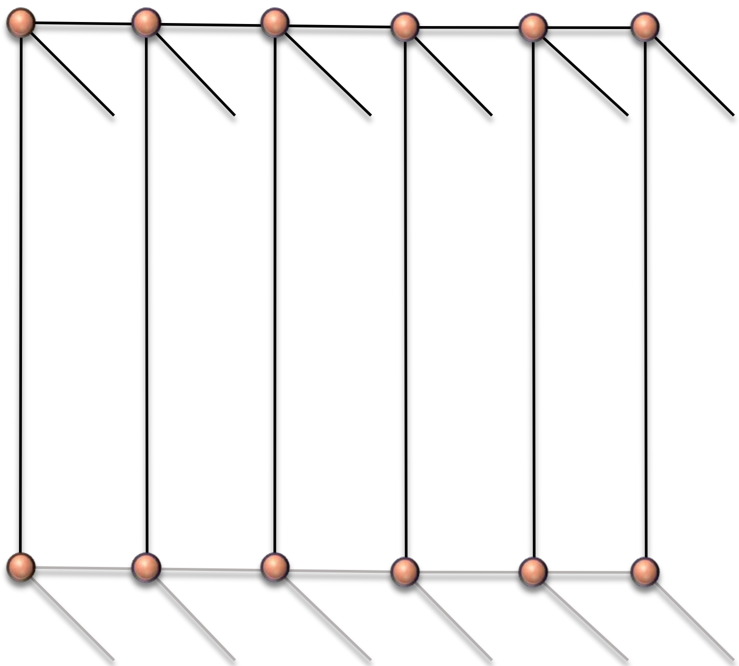
$$\frac{\partial}{\partial A^{*i}}\left(\langle\Psi|H|\Psi\rangle - \lambda\langle\Psi|\Psi\rangle\right) = 0 \quad \Rightarrow \quad \mathbf{H}^i_{eff}\vec{A}^i = \lambda\mathbf{N}^i\vec{A}^i$$

$$\frac{\partial}{\partial A^{*i}}\langle\Psi|\Psi\rangle = \mathbf{N}^i\vec{A}^i$$



e.g. calculation of $\mathbf{N}^i \vec{A}^i$

$$\frac{\partial}{\partial A^{*i}}\left(\langle\Psi|H|\Psi\rangle - \lambda\langle\Psi|\Psi\rangle\right) = 0 \quad \Rightarrow \quad \mathbf{H}_{eff}^i \vec{A}^i = \lambda \mathbf{N}^i \vec{A}^i$$



$$\frac{\partial}{\partial A^{*i}}\langle\Psi|\Psi\rangle = \mathbf{N}^i \vec{A}^i$$

e.g. calculation of $\mathbf{N}^i \vec{A}^i$

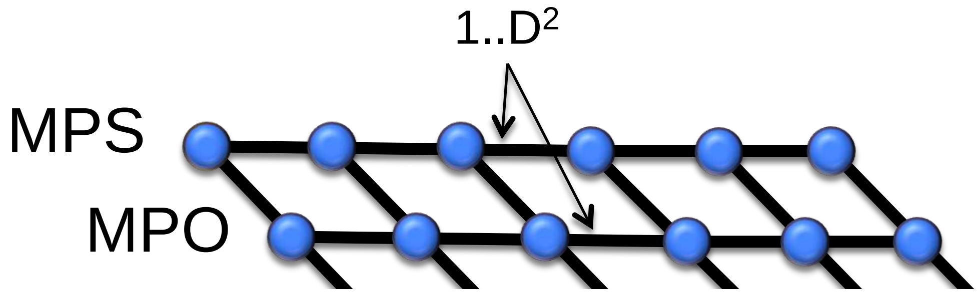
$$\frac{\partial}{\partial A^{*i}}\left(\langle\Psi|H|\Psi\rangle-\lambda\langle\Psi|\Psi\rangle\right)=0 \quad \Rightarrow \quad \mathbf{H}^i_{eff}\vec{A}^i=\lambda\mathbf{N}^i\vec{A}^i$$

MPS

$$\frac{\partial}{\partial A^{*i}}\langle\Psi|\Psi\rangle=\mathbf{N}^i\vec{A}^i$$

e.g. calculation of $\mathbf{N}^i \vec{A}^i$

$$\frac{\partial}{\partial A^{*i}}\left(\langle\Psi|H|\Psi\rangle - \lambda\langle\Psi|\Psi\rangle\right) = 0 \quad \Rightarrow \quad \mathbf{H}^i_{eff}\vec{A}^i = \lambda \mathbf{N}^i \vec{A}^i$$



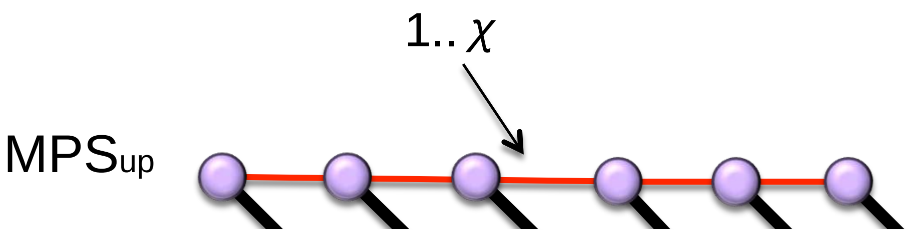
$$\frac{\partial}{\partial A^{*i}}\langle\Psi|\Psi\rangle = \mathbf{N}^i \vec{A}^i$$

1d problem: use a 1d method for MPS (e.g., DMRG or TEBD)

e.g. calculation of $\mathbf{N}^i \vec{A}^i$

$$\frac{\partial}{\partial A^{*i}}\left(\langle\Psi|H|\Psi\rangle - \lambda\langle\Psi|\Psi\rangle\right) = 0 \quad \Rightarrow \quad \mathbf{H}^i_{eff}\vec{A}^i = \lambda \mathbf{N}^i \vec{A}^i$$

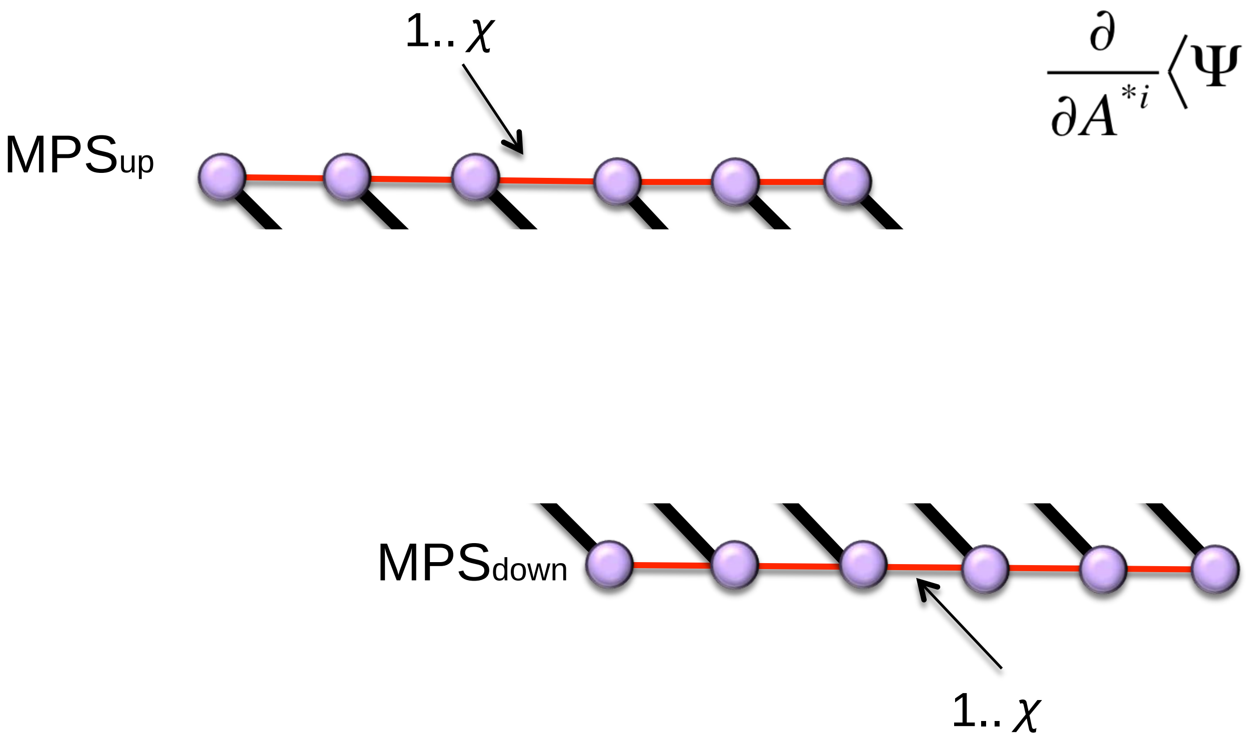
$$\frac{\partial}{\partial A^{*i}}\langle\Psi|\Psi\rangle = \mathbf{N}^i \vec{A}^i$$



e.g. calculation of $\mathbf{N}^i \vec{A}^i$

$$\frac{\partial}{\partial A^{*i}}\left(\langle\Psi|H|\Psi\rangle - \lambda\langle\Psi|\Psi\rangle\right) = 0 \quad \Rightarrow \quad \mathbf{H}^i_{eff}\vec{A}^i = \lambda \mathbf{N}^i \vec{A}^i$$

$$\frac{\partial}{\partial A^{*i}}\langle\Psi|\Psi\rangle = \mathbf{N}^i \vec{A}^i$$

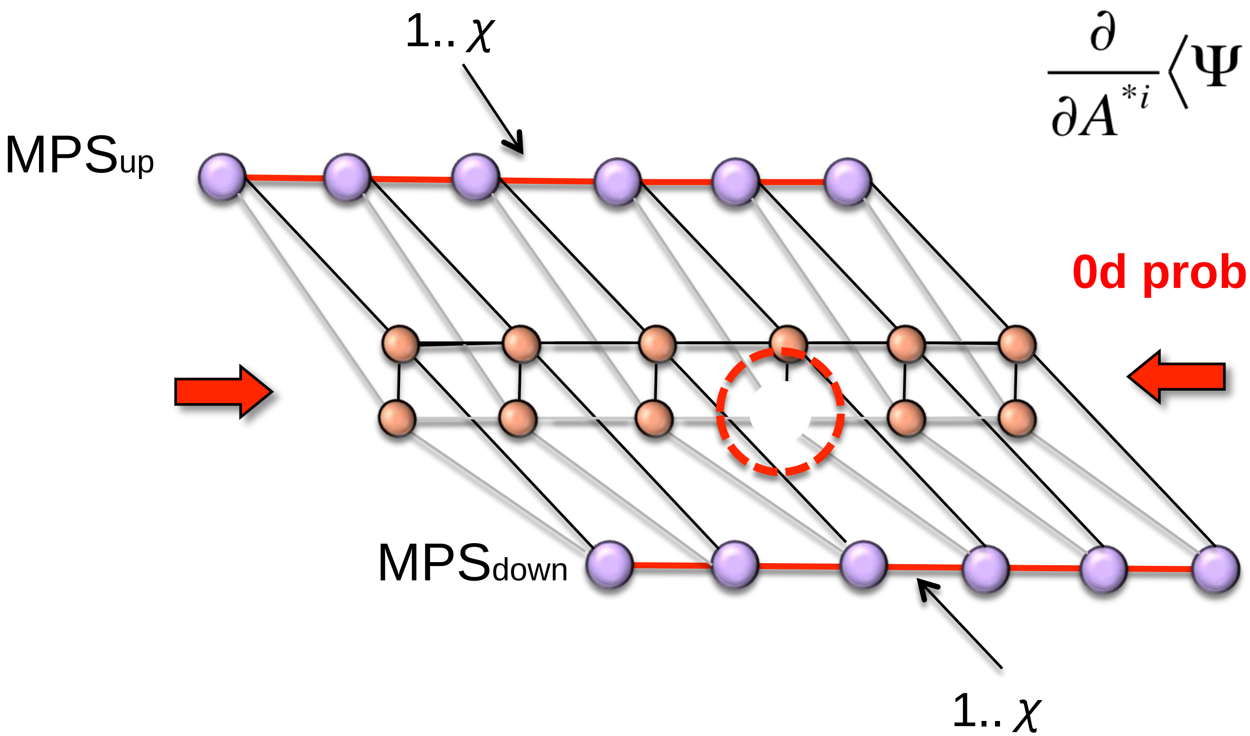


e.g. calculation of $\mathbf{N}^i \vec{A}^i$

$$\frac{\partial}{\partial A^{*i}}\left(\langle\Psi|H|\Psi\rangle - \lambda\langle\Psi|\Psi\rangle\right) = 0 \quad \Rightarrow \quad \mathbf{H}^i_{eff}\vec{A}^i = \lambda \mathbf{N}^i \vec{A}^i$$

$$\frac{\partial}{\partial A^{*i}}\langle\Psi|\Psi\rangle = \mathbf{N}^i \vec{A}^i$$

0d problem: exact!



e.g. calculation of $\mathbf{N}^i \vec{A}^i$

$$\frac{\partial}{\partial A^{*i}}\Big(\langle\Psi|H|\Psi\rangle - \lambda\langle\Psi|\Psi\rangle\Big) = 0 \quad \Rightarrow \quad \mathbf{H}^i_{eff}\vec{A}^i = \lambda \mathbf{N}^i \vec{A}^i$$

$$\frac{\partial}{\partial A^{*i}}\langle\Psi|\Psi\rangle = \mathbf{N}^i \vec{A}^i$$

Dimensional reduction

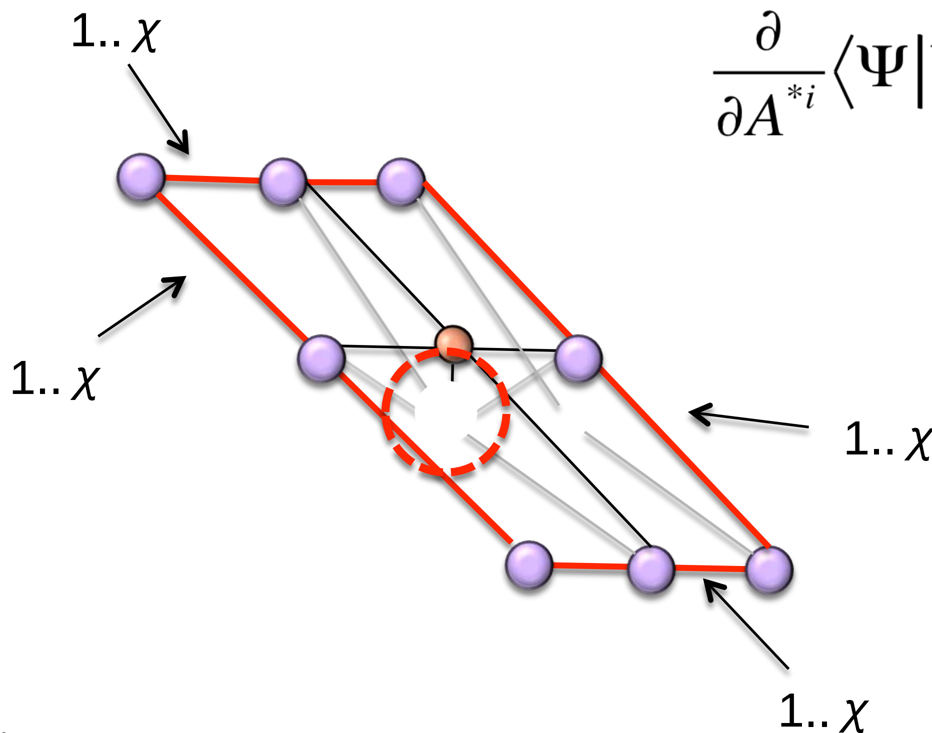
2d problem

⬇

1d problem: use DMRG!

⬇

0d problem: exact!



$\mathbf{N}^i \vec{A}^i$ is the ***environment*** of tensor A^{*i}

$\mathbf{H}^i_{eff}\vec{A}^i$ is computed similarly, but sandwitching with the Hamiltonian

Valid also for any expectation value

Time evolution

(real, imaginary)

Time evolution (e.g. imaginary)

e.g. J. Jordan et al, PRL 101, 250602 (2008)

$$|\Psi_0\rangle = \lim_{\tau\to\infty} \frac{e^{-\tau H}|\Psi\rangle}{\left\| e^{-\tau H}|\Psi\rangle \right\|}$$

Divide into small time-steps $\delta\tau << 1$

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Divide into small time-steps $\delta\tau << 1$

$$\begin{array}{ccccccccccc}
\left|\Psi\right\rangle & \xrightarrow{e^{-\delta\tau H}} & \left|\Psi^1\right\rangle & \xrightarrow{e^{-\delta\tau H}} & \cdots & \xrightarrow{e^{-\delta\tau H}} & \left|\Psi^m\right\rangle & \xrightarrow{e^{-\delta\tau H}} & \left|\Psi^{m+1}\right\rangle & \xrightarrow{e^{-\delta\tau H}} & \cdots \\
E & \geq & E^1 & \geq & \cdots & \geq & E^m & \geq & E^{m+1} & \geq & \cdots
\end{array}$$

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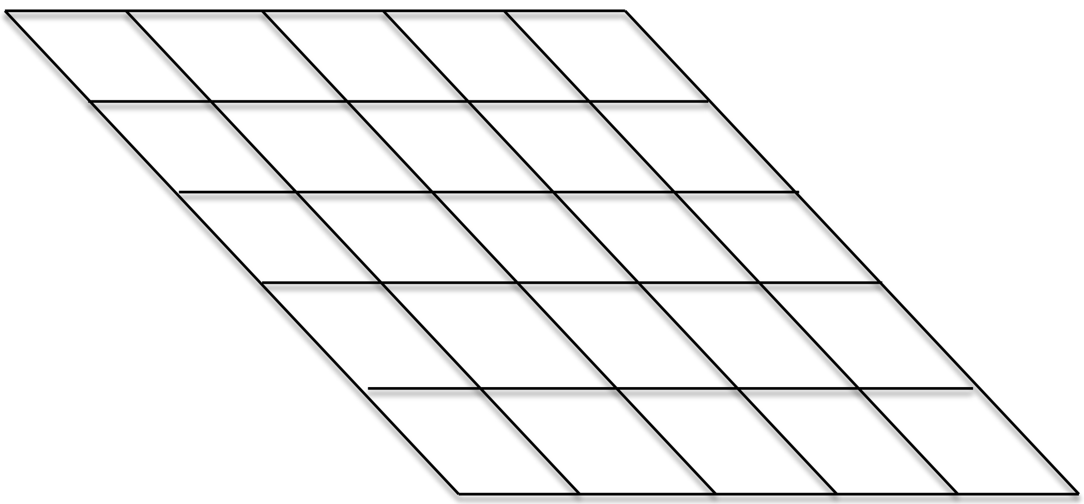
$$E \geq E^1 \geq \cdots \geq E^m \geq E^{m+1} \geq \cdots$$

Split the Hamiltonian
(e.g. 2-body n.n.)

$$H = H_{hor}^{even} + H_{hor}^{odd} + H_{ver}^{even} + H_{ver}^{odd}$$



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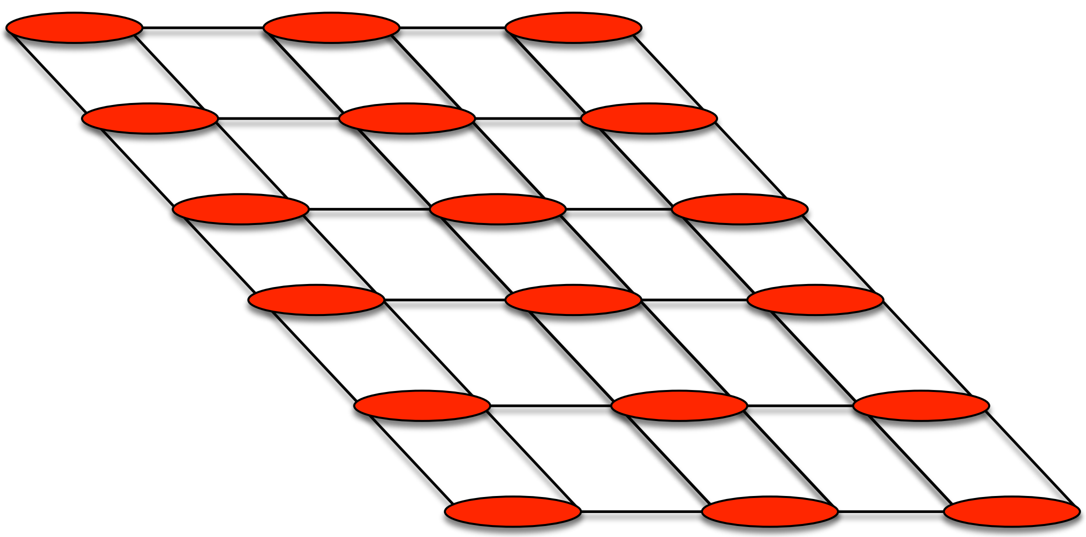

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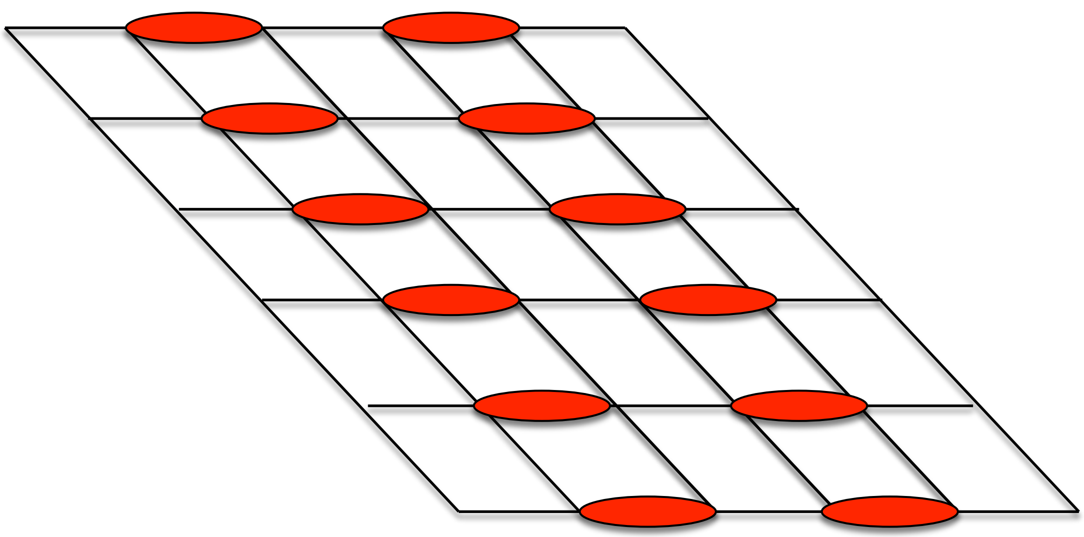

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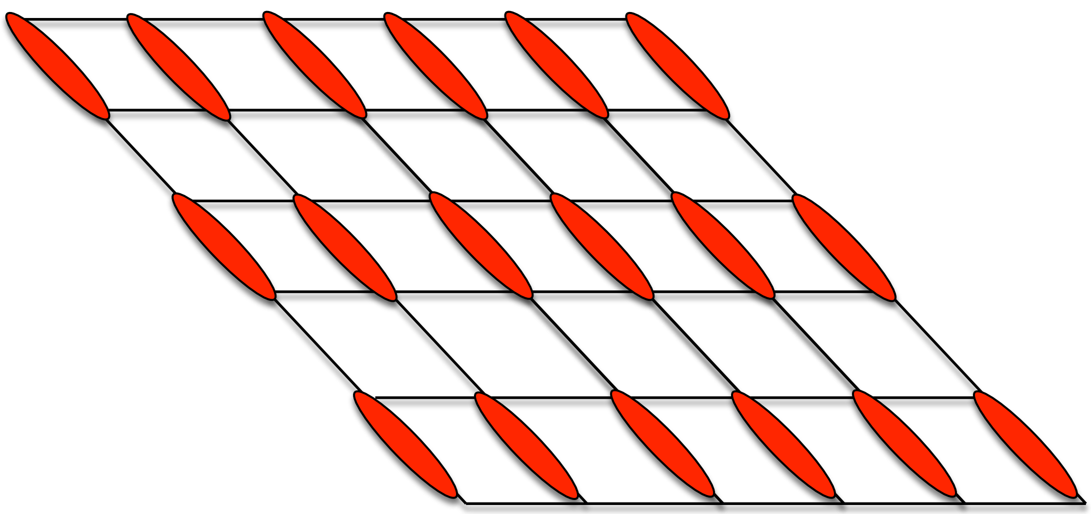

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Divide into small time-steps $\delta\tau << 1$

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$$E \geq E^1 \geq \cdots \geq E^m \geq E^{m+1} \geq \cdots$$

Split the Hamiltonian
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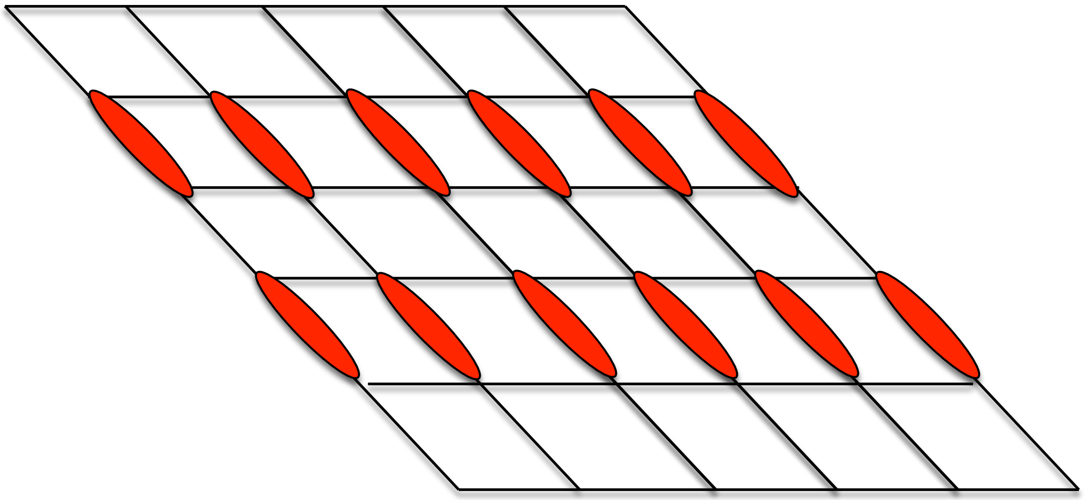
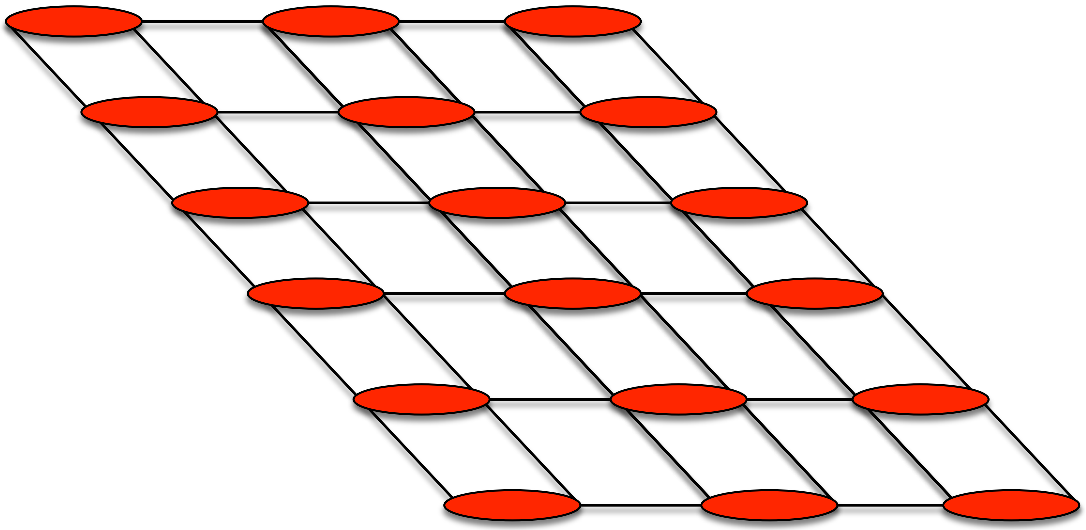


Time evolution (e.g. imaginary)

e.g. J. Jordan et al, PRL 101, 250602 (2008)

$$e^{-\delta\tau H} \approx e^{-\delta\tau H_{hor}^{even}} e^{-\delta\tau H_{hor}^{odd}} e^{-\delta\tau H_{ver}^{even}} e^{-\delta\tau H_{ver}^{odd}} + O(\delta\tau^2)$$

$$e^{-\delta\tau H_{hor}^{even}}$$



Time evolution (e.g. imaginary)

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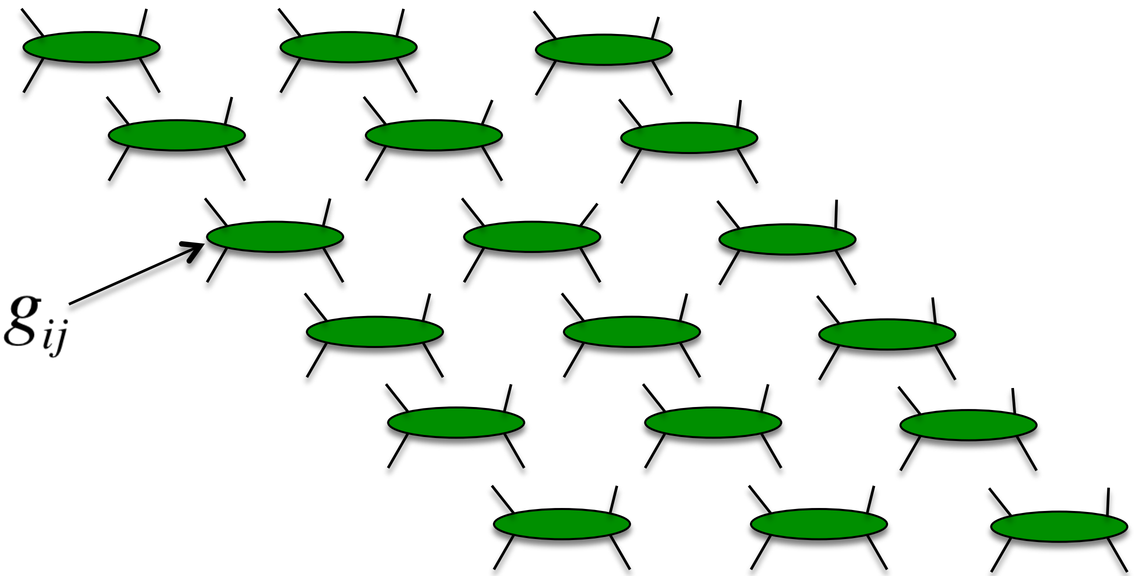
$$e^{-\delta\tau H} \approx e^{-\delta\tau H_{hor}^{even}} e^{-\delta\tau H_{hor}^{odd}} e^{-\delta\tau H_{ver}^{even}} e^{-\delta\tau H_{ver}^{odd}} + O(\delta\tau^2)$$

$$e^{-\delta\tau H_{hor}^{even}} = \bigotimes_{<i,j>} e^{-\delta\tau h_{ij}} = \bigotimes_{<i,j>} g_{ij}$$

2-body gates



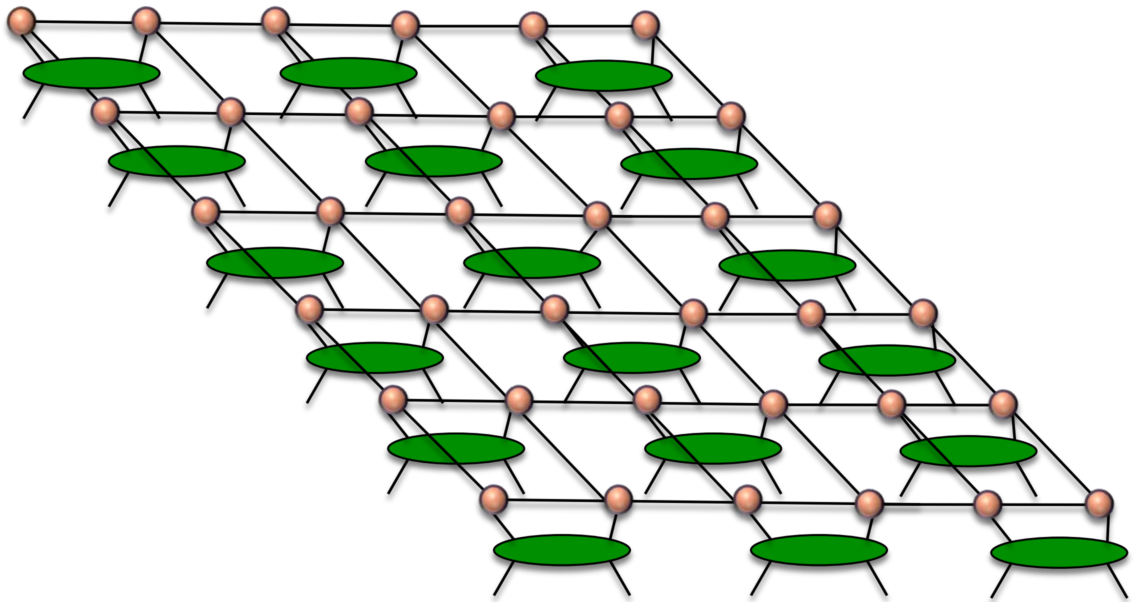
Time evolution (e.g. imaginary)

e.g. J. Jordan et al, PRL 101, 250602 (2008)

$$e^{-\delta\tau H} \approx e^{-\delta\tau H_{hor}^{even}} e^{-\delta\tau H_{hor}^{odd}} e^{-\delta\tau H_{ver}^{even}} e^{-\delta\tau H_{ver}^{odd}} + O(\delta\tau^2)$$

$$e^{-\delta\tau H_{hor}^{even}} \left|\Psi\right\rangle = \left|\tilde{\Psi}\right\rangle$$

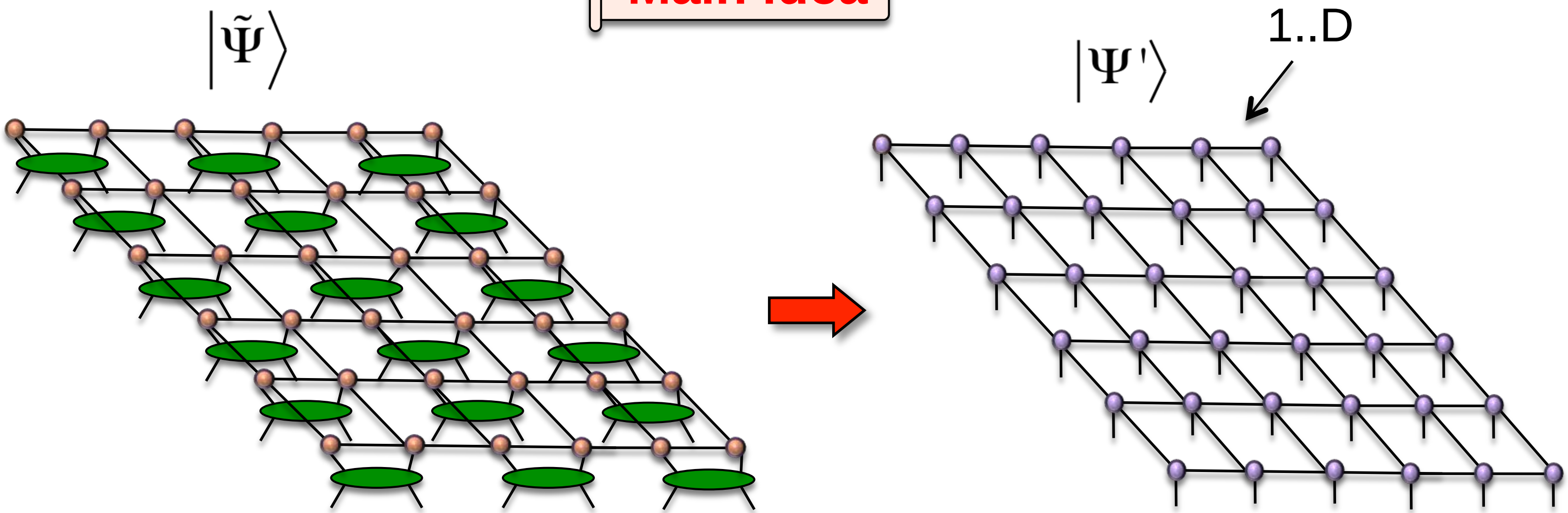
evolved state



Time evolution (e.g. imaginary)

e.g. J. Jordan et al, PRL 101, 250602 (2008)

Main idea



Different approaches to this problem: (fast) full update, simplified update, TPVA...

Full update: $\min \left\| |\tilde{\Psi}\rangle - |\Psi'\rangle \right\|^2$

Finite systems: optimize over all tensors in the PEPS (as before)

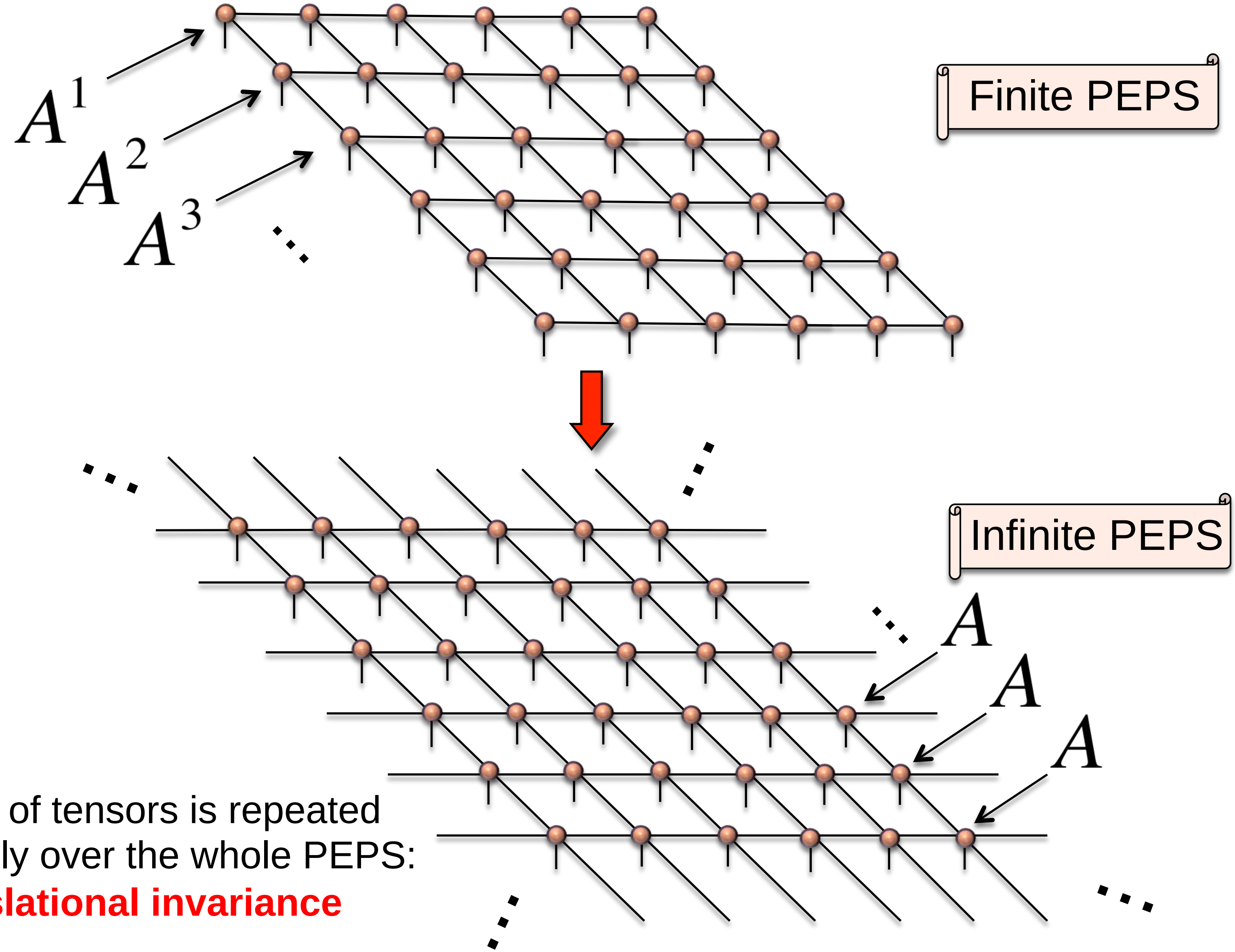
Infinite systems: optimize over tensors in the PEPS unit cell (iPEPS)

Require ***calculations of environments***, like the one shown before.

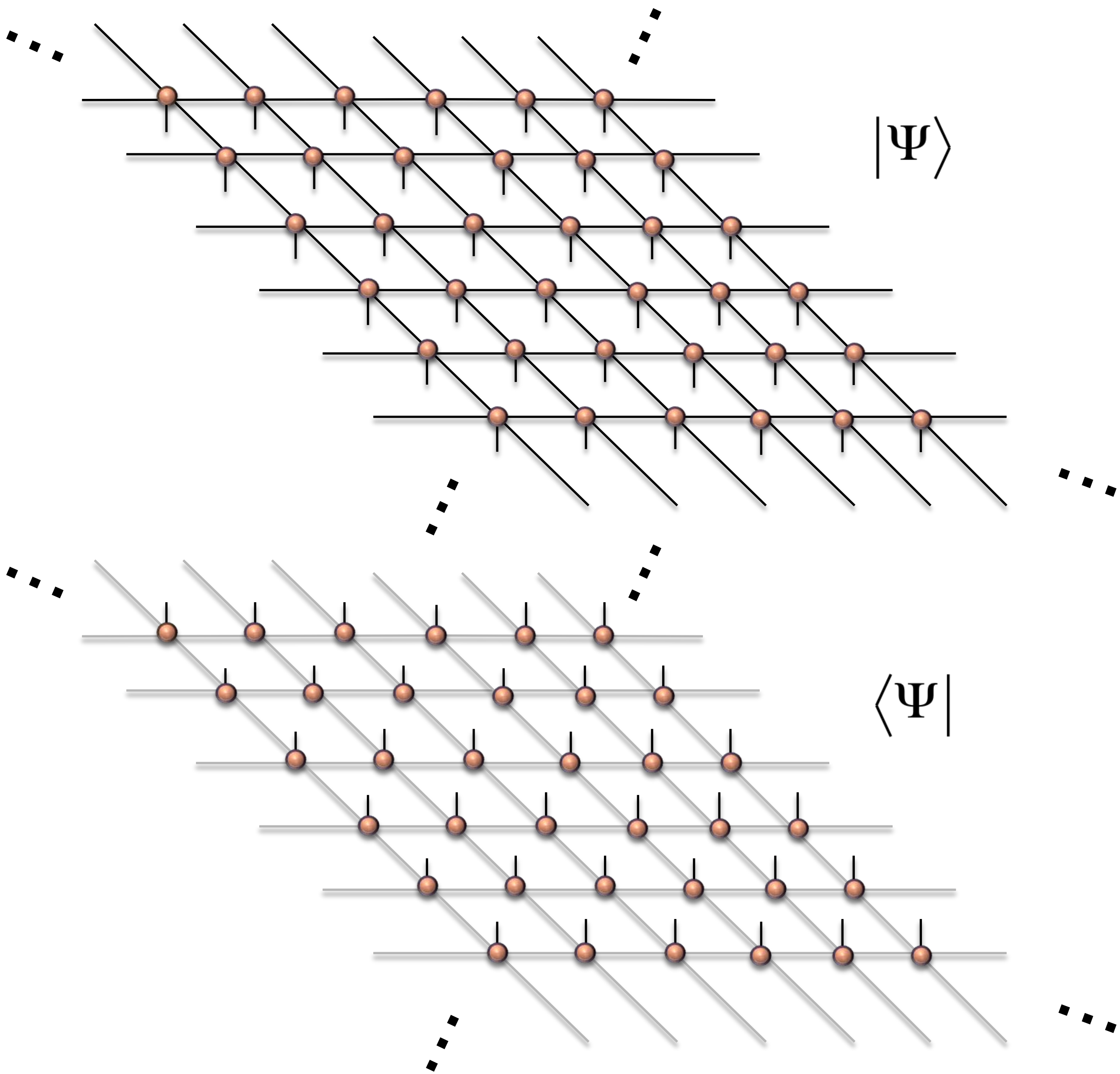
Environments with infinite PEPS

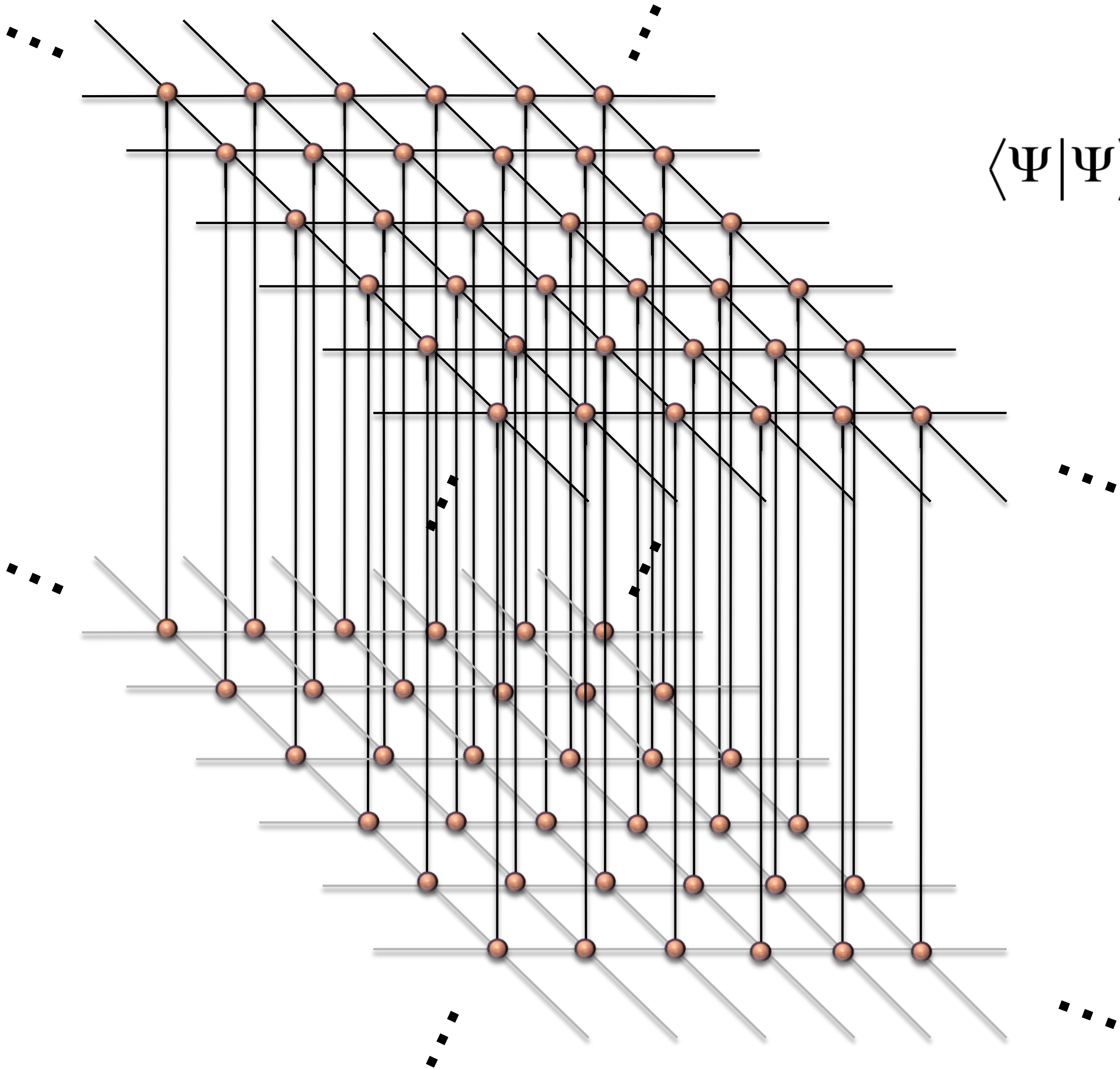
Infinite systems

e.g. J. Jordan et al, PRL 101, 250602 (2008)
R. Orús, G. Vidal, PRB 80 094403 (2009)

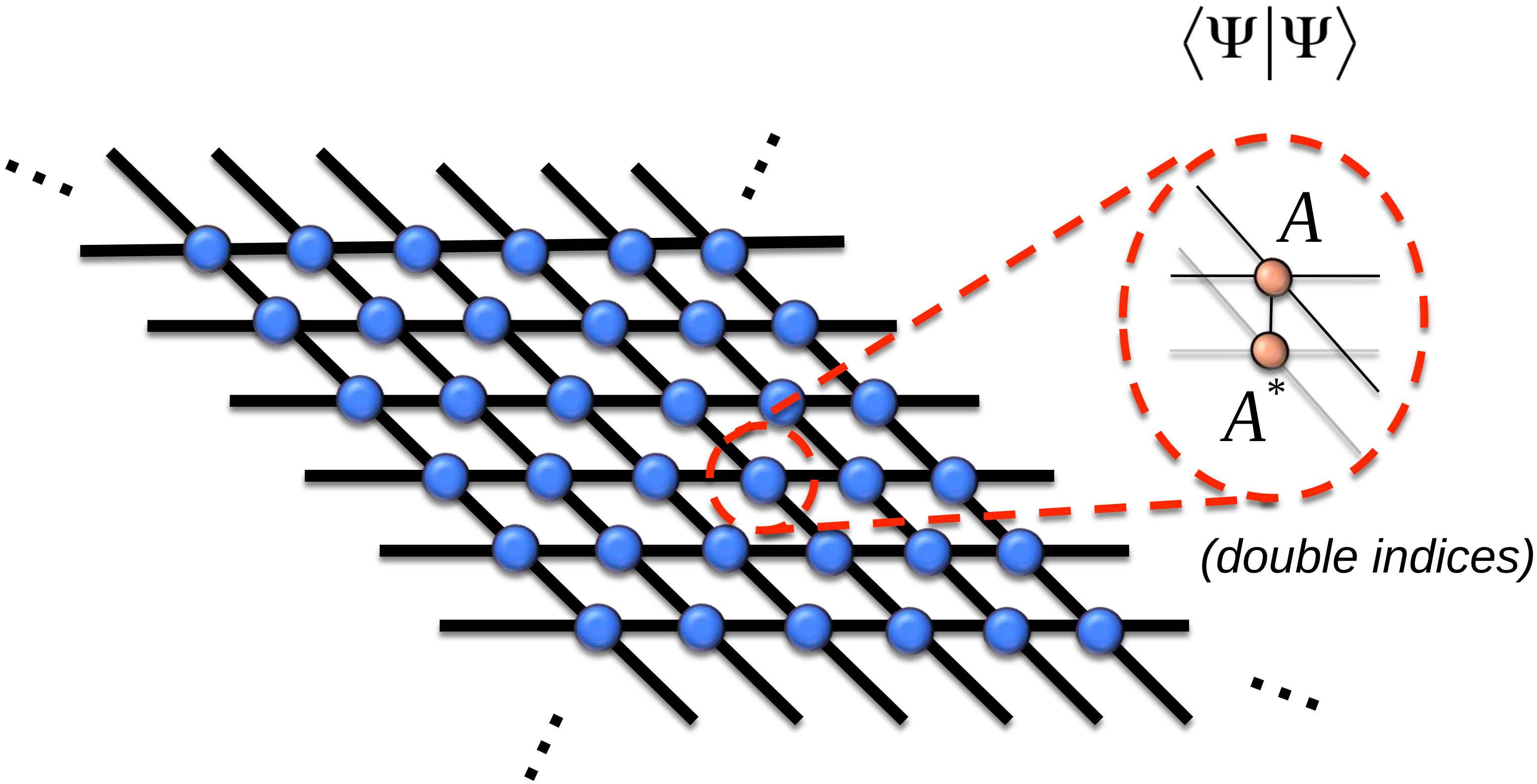


Unit cell of tensors is repeated periodically over the whole PEPS:
translational invariance

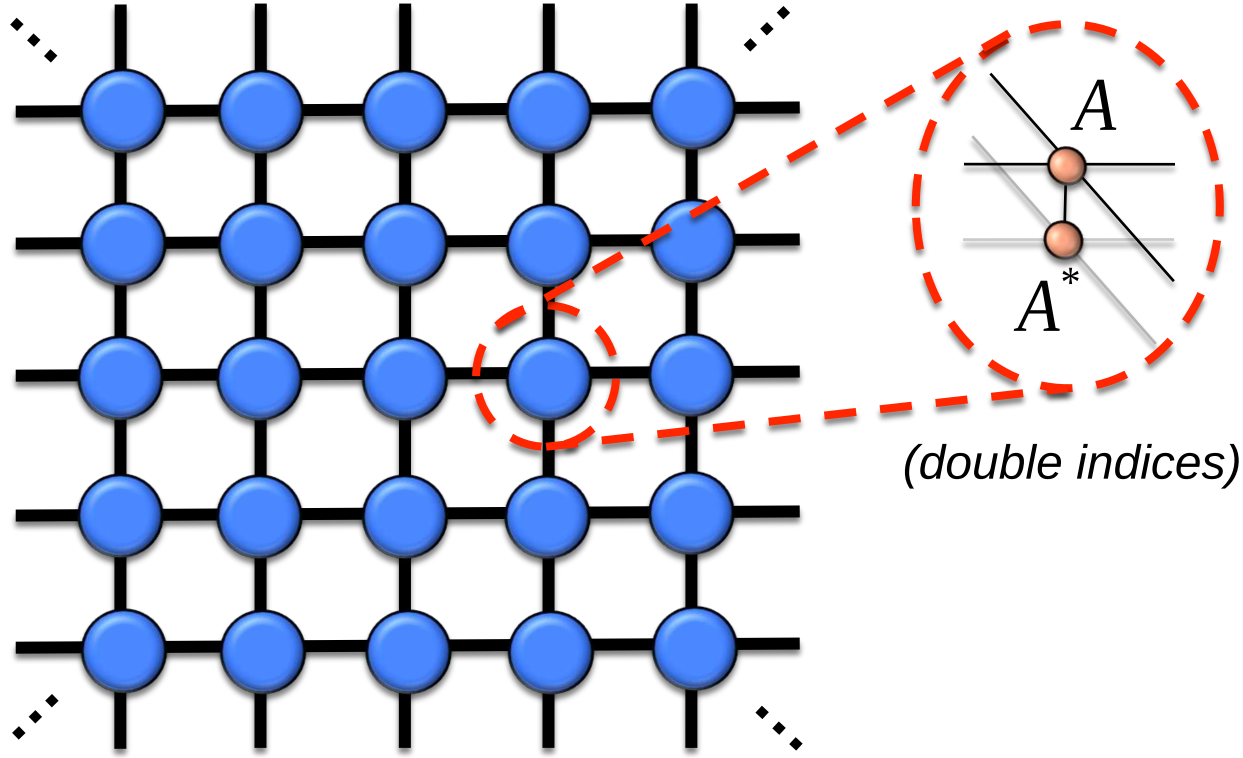

$|\Psi\rangle$
$\langle\Psi|$



$$\langle\Psi|\Psi\rangle$$

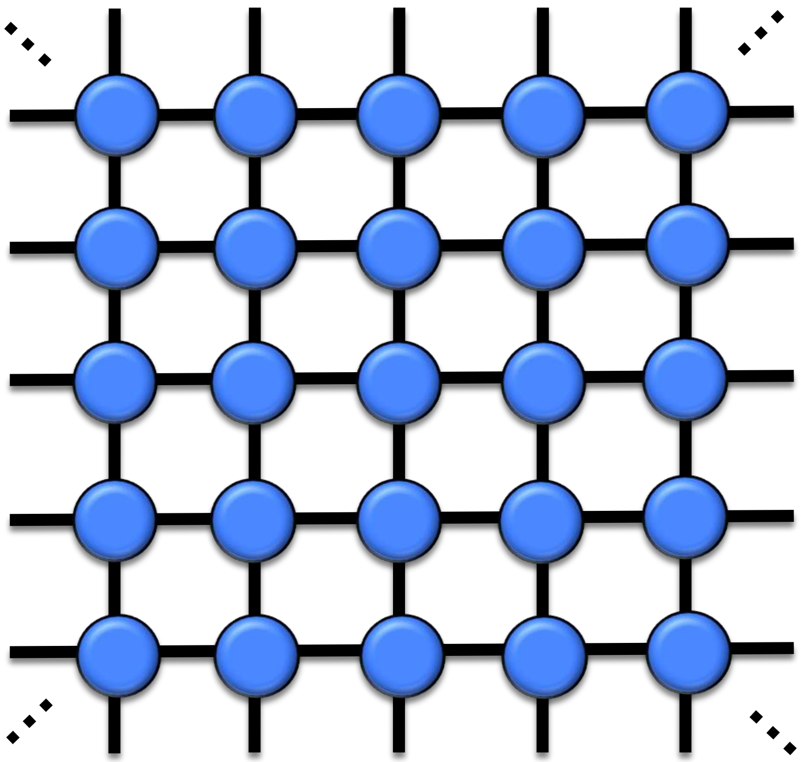


Let's put it on the plane of the screen!

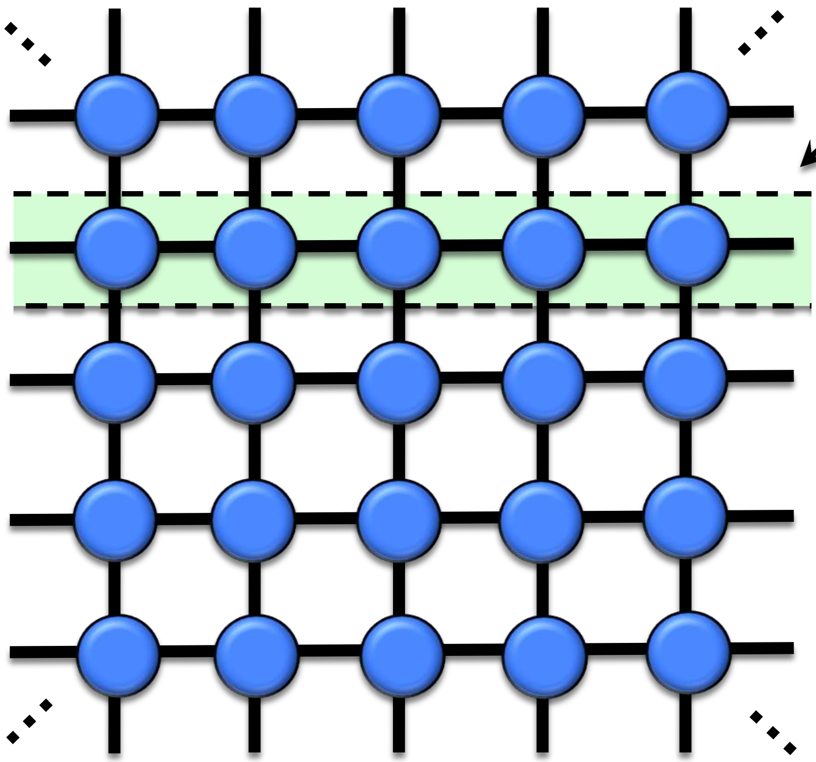


Environment calculations ⟷ ***Contraction of this infinite lattice***

Contracting the infinite 2d lattice

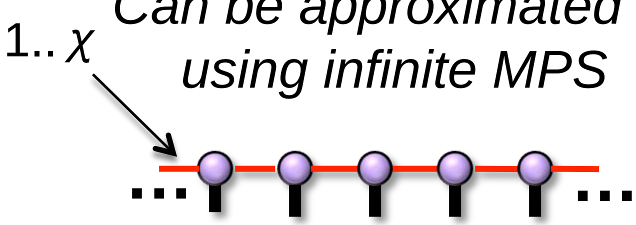


Contracting the infinite 2d lattice



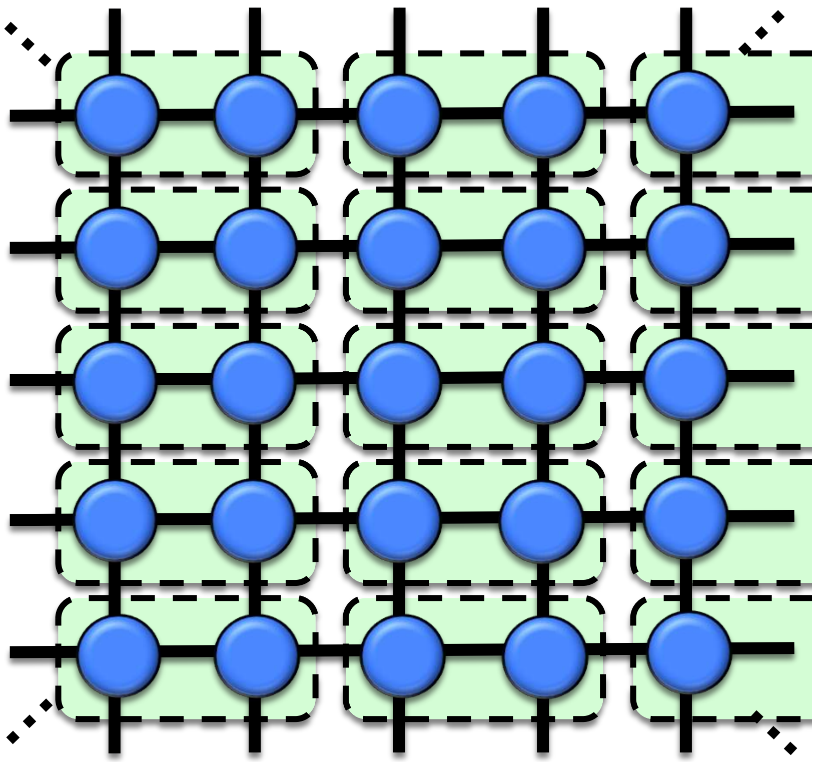
1-dim transfer matrix:
dominant eigenvector?

Can be approximated using infinite MPS



iTEBD, iDMRG, PWFRG, etc

Contracting the infinite 2d lattice

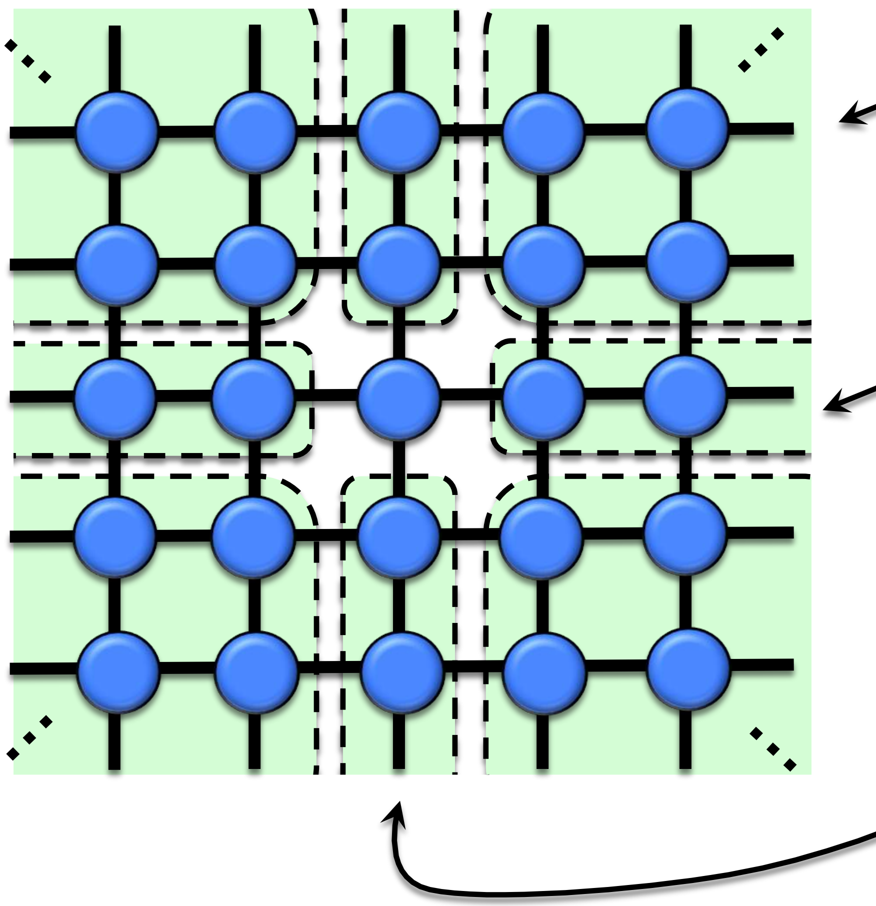


Coarse-graining approaches:

TRG/SRG, HOSVD, TNR

c.f. T. Xiang's & G. Evenbly's talks

Contracting the infinite 2d lattice



corner transfer matrix *(Baxter, 1968)*

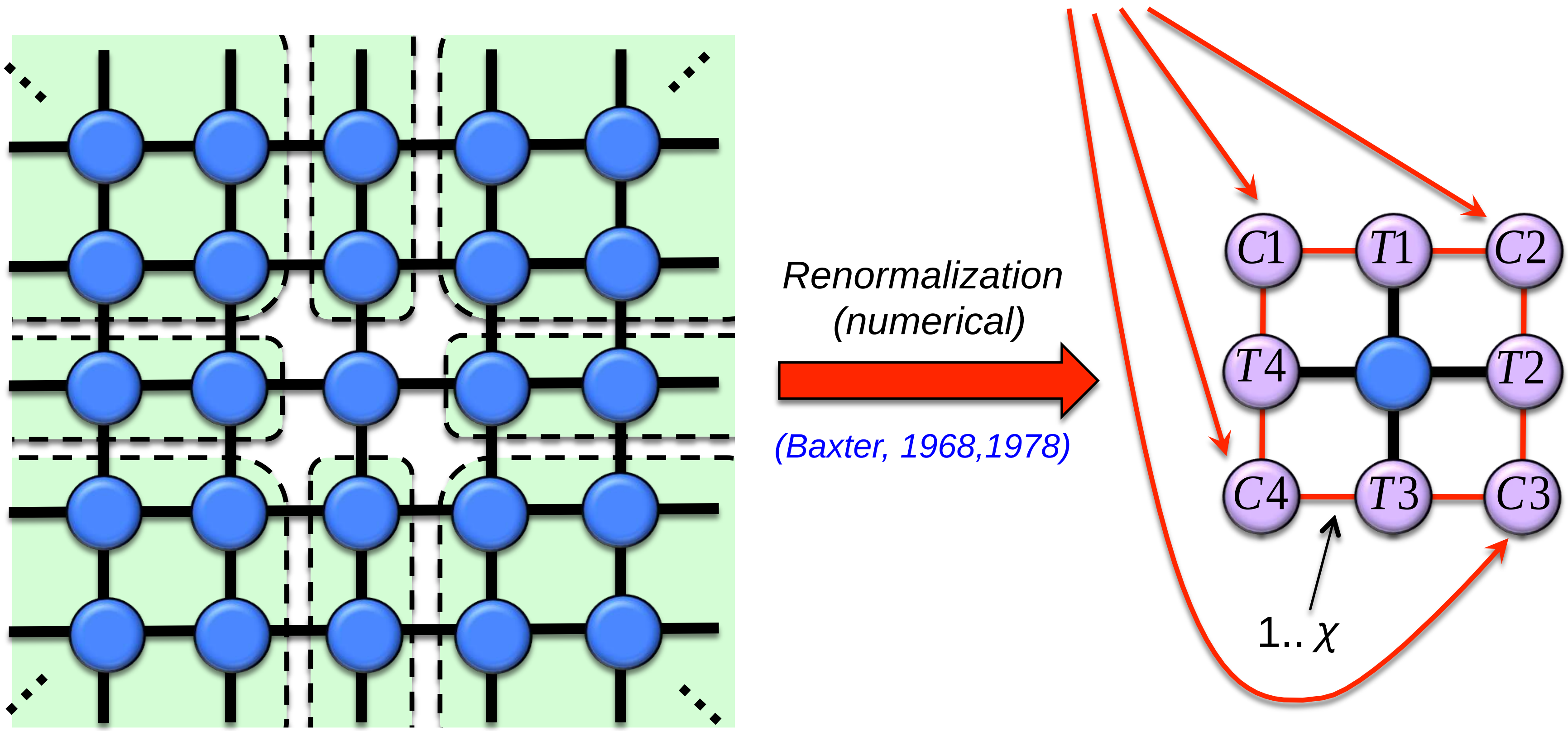
(nice spectral properties)

half-row transfer matrix

half-column transfer matrix

Contracting the infinite 2d lattice

Renormalized Corner Transfer Matrices



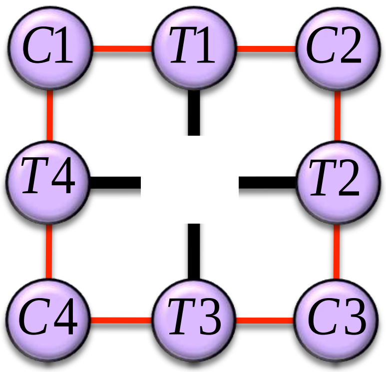
Directional version *of the* ***corner transfer matrix renormalization group***

(faster than 1d transfer matrix methods) *T. Nishino, K. Okunishi, JPS Jpn. 65, 891 (1996)*

Corner transfer matrix and iPEPS

R. Orús, G. Vidal, PRB 80 094403 (2009), R. Orús, PRB 85, 205117 (2012)

Example: left move



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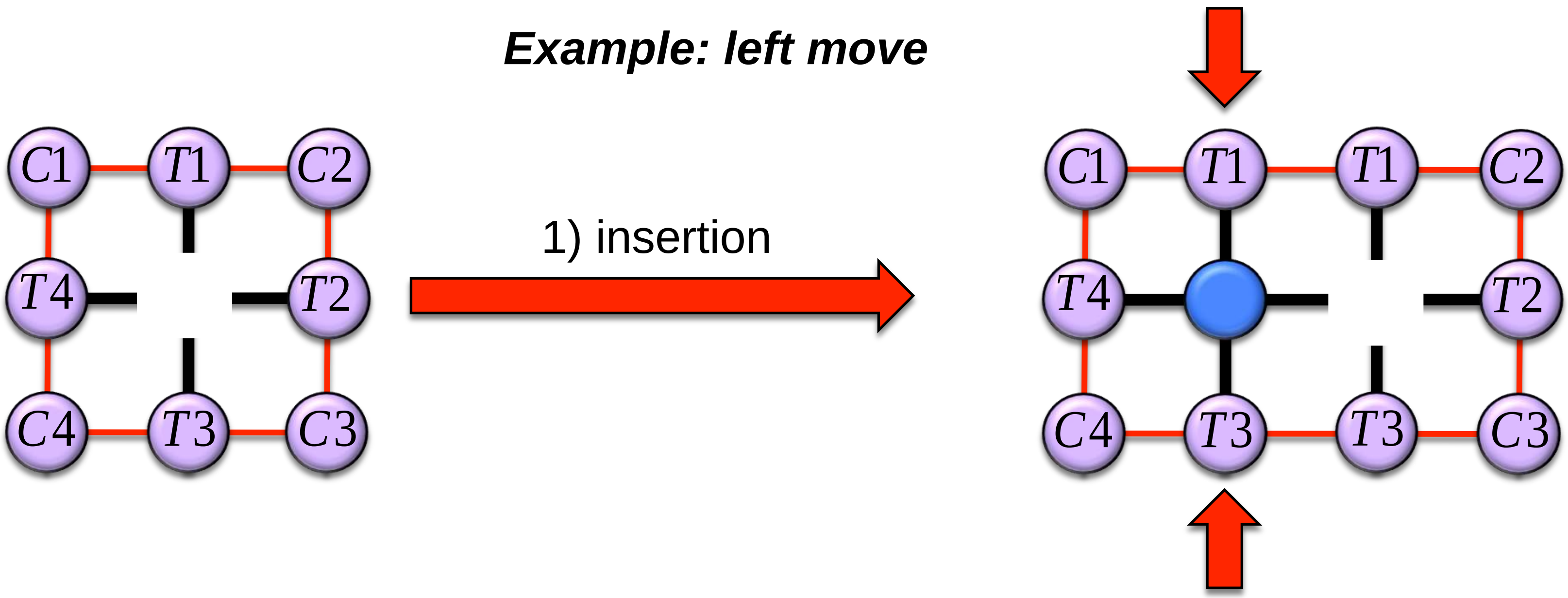

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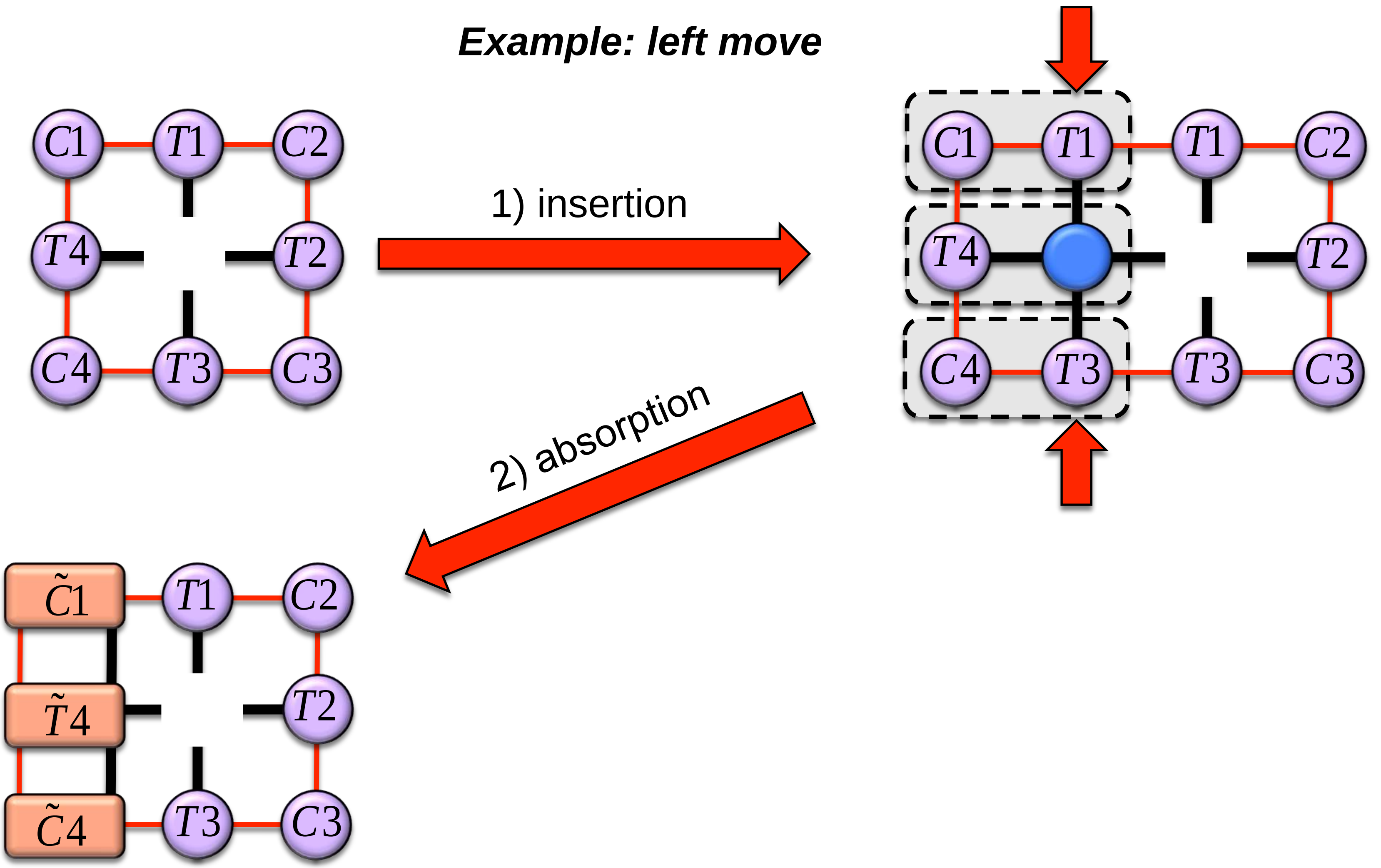

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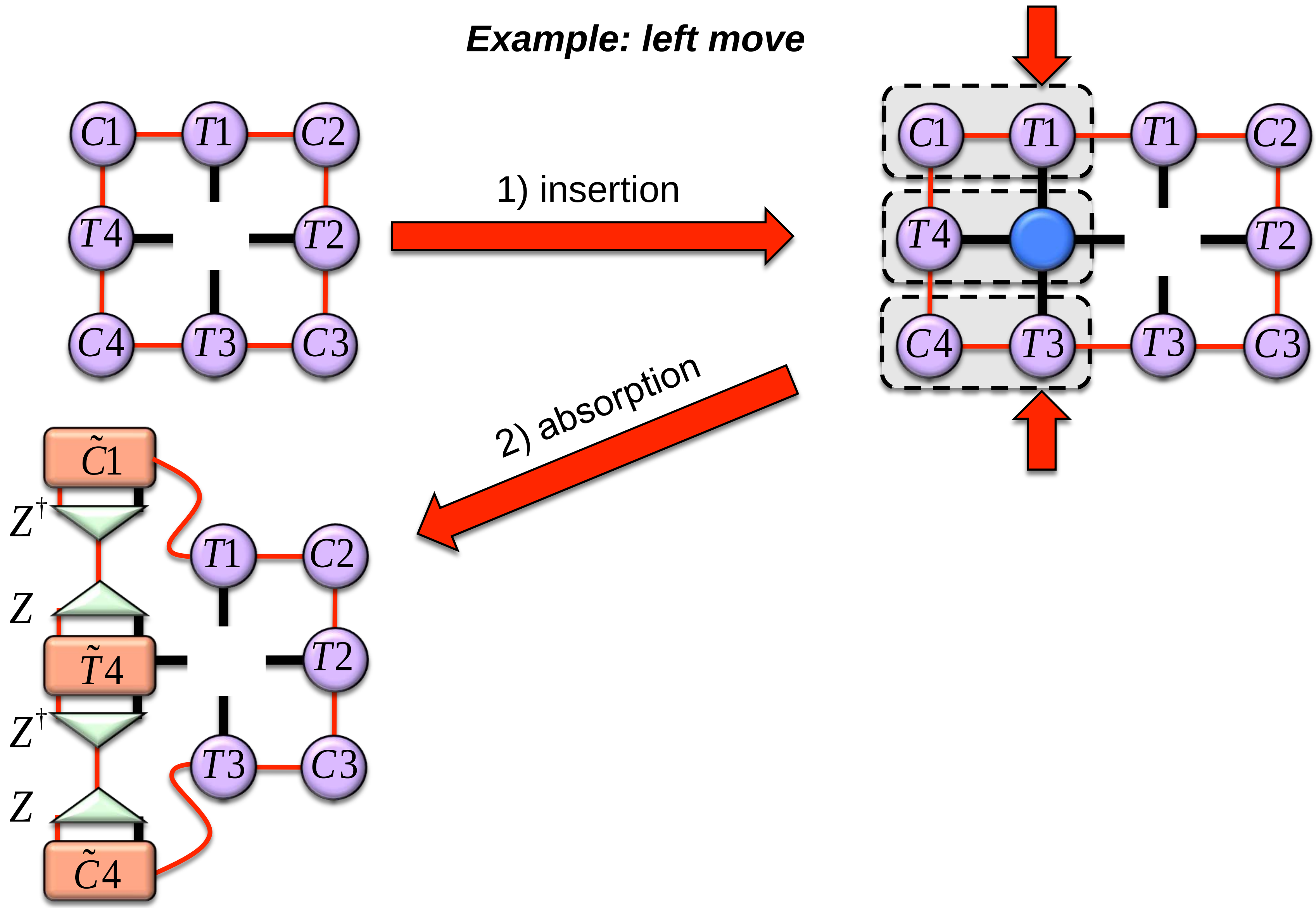
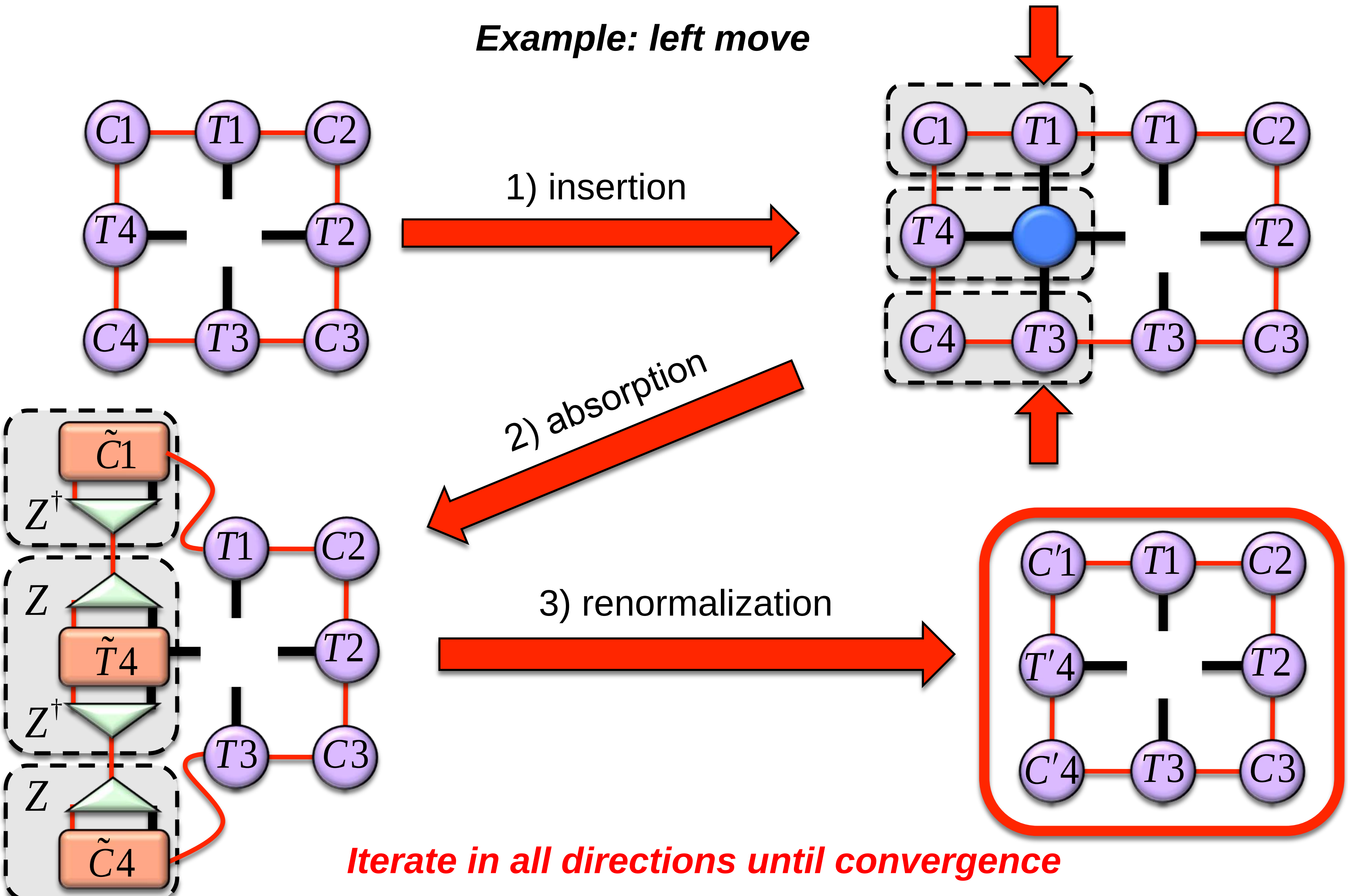


Corner transfer matrix and iPEPS

R. Orús, G. Vidal, PRB 80 094403 (2009), R. Orús, PRB 85, 205117 (2012)

Example: left move

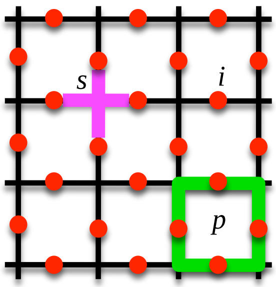


Iterate in all directions until convergence

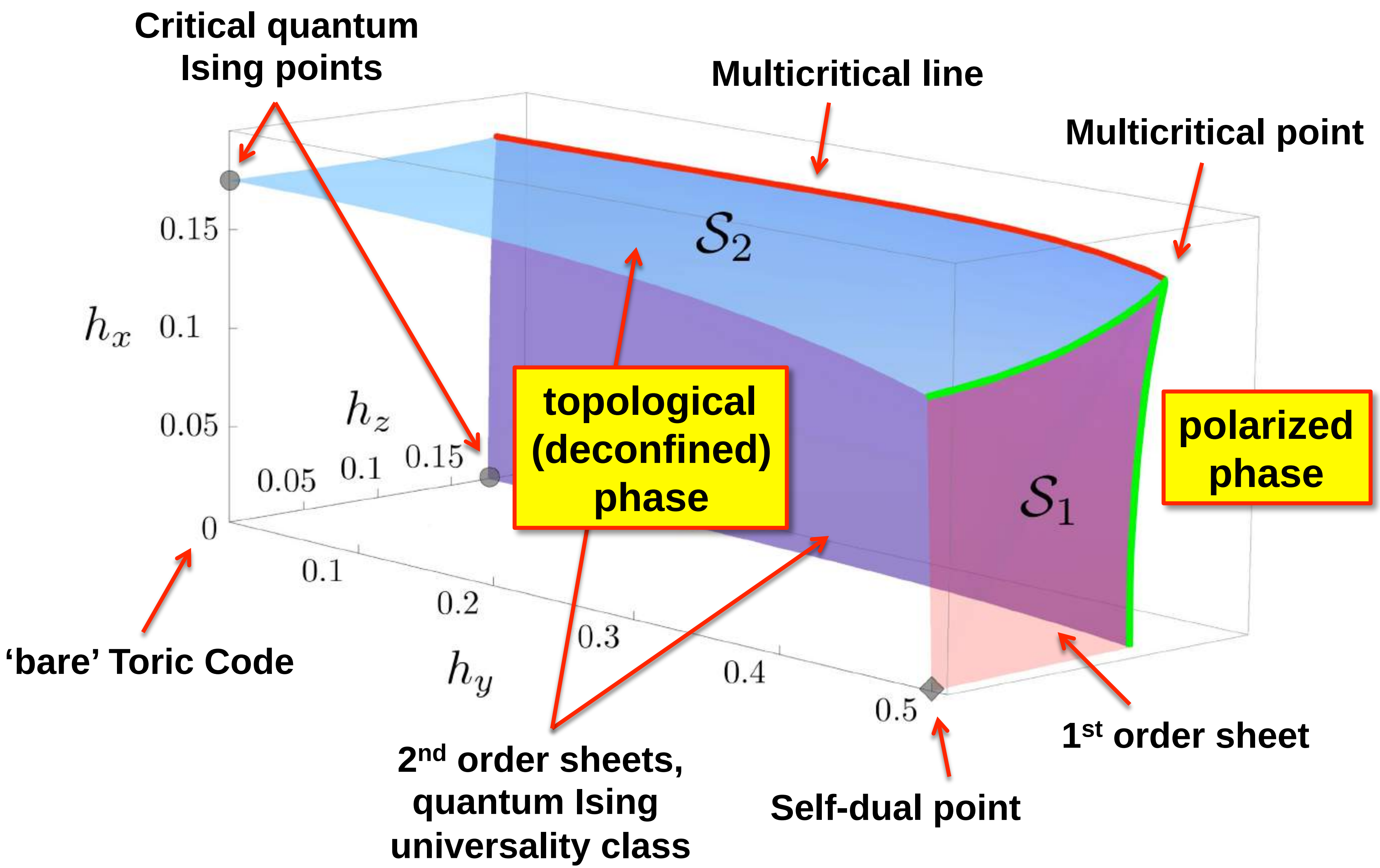
A typical example:
Toric Code in arbitrary field

S. Dusuel et al., PRL 106, 107203 (2011)

$$H = -J\sum_s A_s - J\sum_p B_p - h_x\sum_i \sigma_i^x - h_y\sum_i \sigma_i^y - h_z\sum_i \sigma_i^z$$



Phase diagram



Outline

1) Entanglement and 1d MPS

2) 2d PEPS

2.1) Basics

2.2) Computing

2.3) Fermions

3) Some further topics

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Fermionic 2d systems

Fermionic systems are extremely interesting physical systems, e.g. the **2d fermionic Hubbard model** may be the key to understand the emergence of **high-T_C superconductivity**

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Unfortunately, fermionic systems are also **amongst the most difficult** to simulate, because of the **sign problem in Quantum MonteCarlo (sampling of negative probabilities)**

Fermions are a NUMERICAL MONSTER

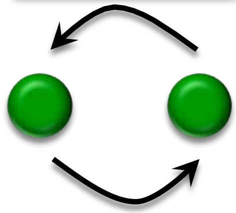
for Quantum MonteCarlo because of the sign problem!

but there is hope...

Tensor Networks + Fermions

e.g., P. Corboz, R. Orús, B. Bauer, G. Vidal, PRB 81, 165104 (2010)

Bosons



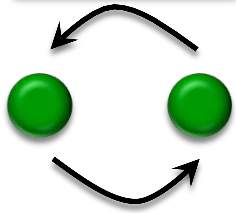
$$\Psi_B(x_1, x_2) = \Psi_B(x_2, x_1)$$

Symmetric wavefunction

$$b_i b_j = b_j b_i$$

Operators commute

Bosons



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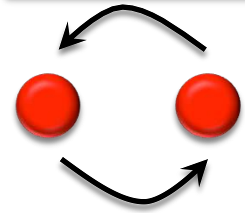


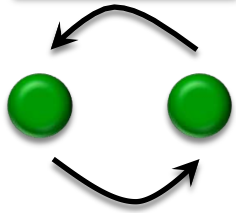
$$\Psi_F(x_1, x_2) = -\Psi_F(x_2, x_1)$$

Antisymmetric wavefunction

$$c_i c_j = -c_j c_i$$

Operators anticommute

Bosons



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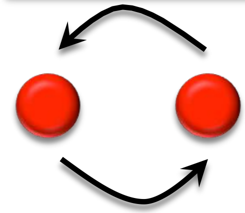

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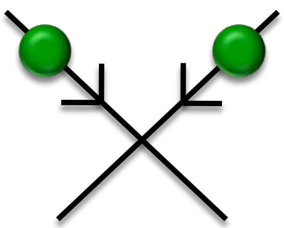
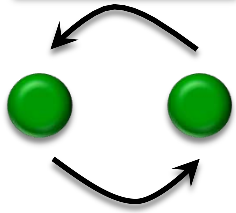
Crossings in a TN



+

Ignore crossings

Bosons



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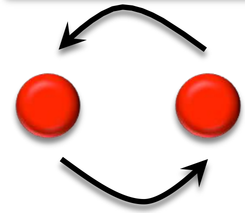

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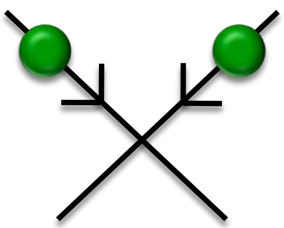

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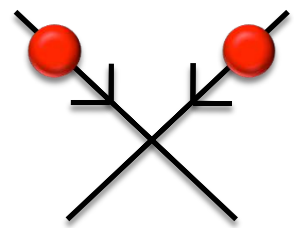

—

Careful!!!!

Tensor Network “fermionization” rules

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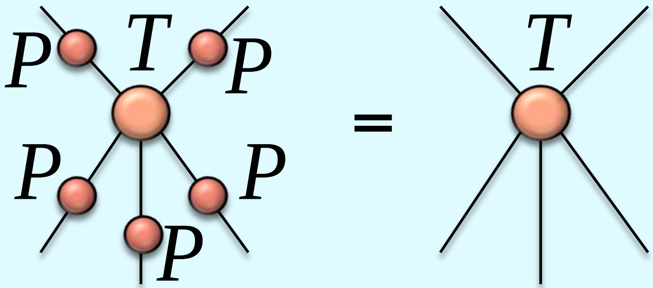
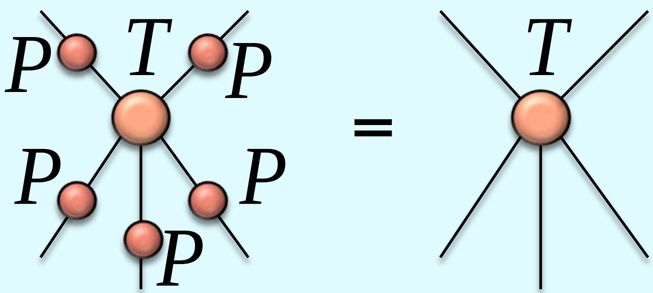


Use ***parity-preserving*** *tensors*

$$T_{i_1 i_2 \ldots i_M} = 0 \ \text{ if } \ P(i_1)P(i_2)\ldots P(i_M) \neq 1$$

Symmetry of the Hamiltonian

Tensor Network "fermionization" rules



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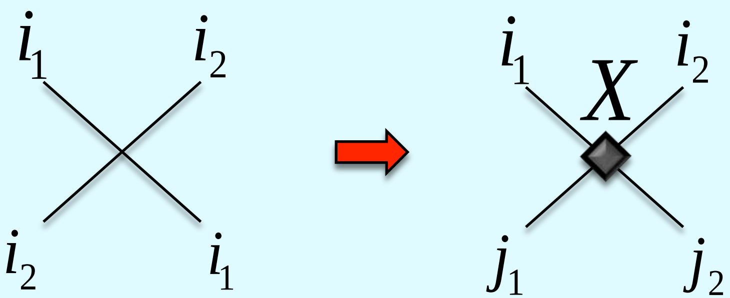


Replace crossings by

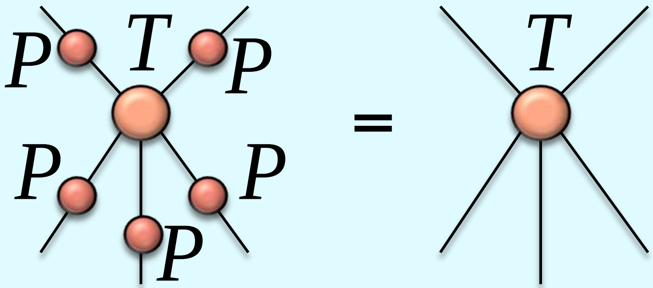
fermionic swap gates

$$X_{i_2 i_1 j_1 j_2} = \delta_{i_1 j_1} \delta_{i_2 j_2} S(P(i_1), P(i_2))$$

$$S(P(i_1), P(i_2)) = \begin{cases} -1 & \text{if } \ P(i_1) = P(i_2) = -1 \\ +1 & \text{otherwise} \end{cases}$$

*Fermionic operators **anti**commute*

Tensor Network "fermionization" rules



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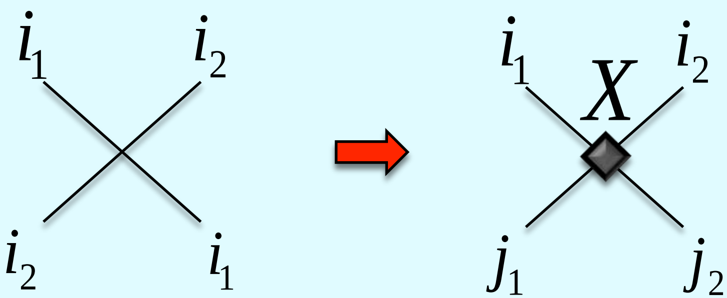

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*Fermionic operators **anti**commute*

The leading order of the computational cost is the same as in the bosonic case

fermionic order ~ graphical projection of a PEPS

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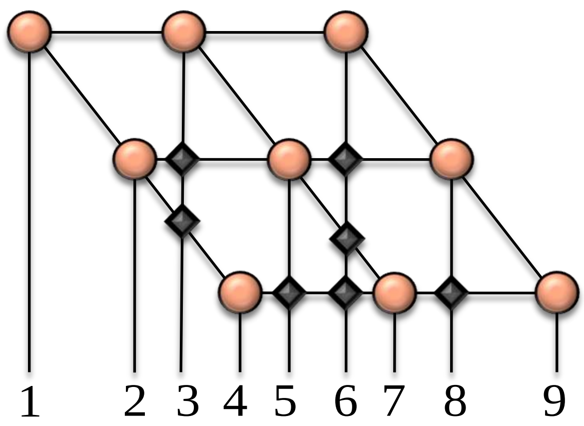
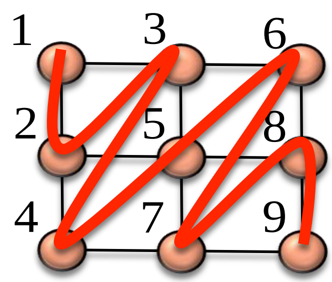
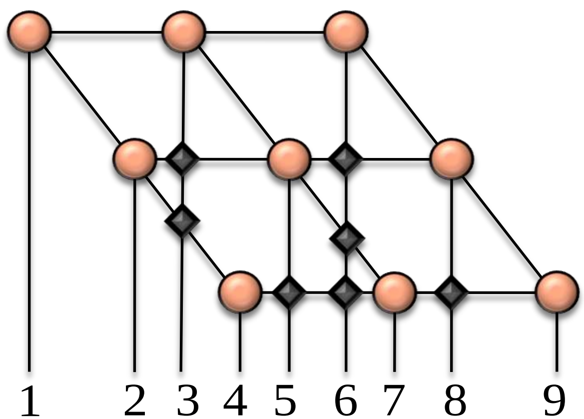
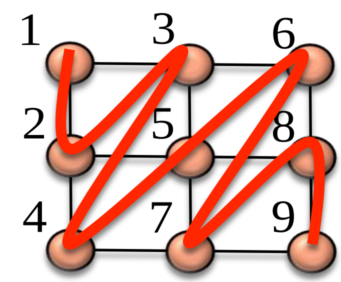
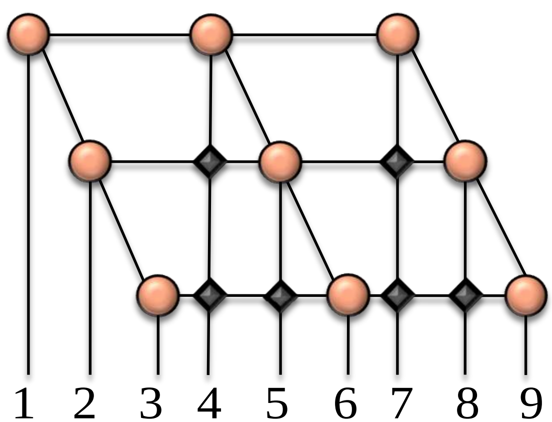


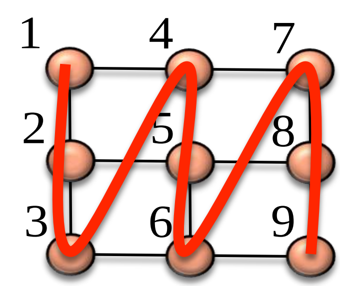


fermionic order ~ graphical projection of a PEPS









physics is independent of the order

⇕

physics is independent of graphical projection

(different choices of Jordan-Wigner transformation, if mapping to a spin system)

Example: *scalar product of 3x3 PEPS*

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$$\langle \Psi | \Psi \rangle$$

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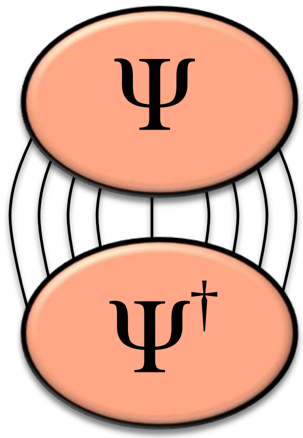
$$\langle \Psi | \Psi \rangle \Rightarrow$$



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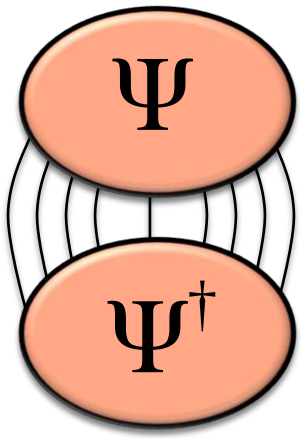
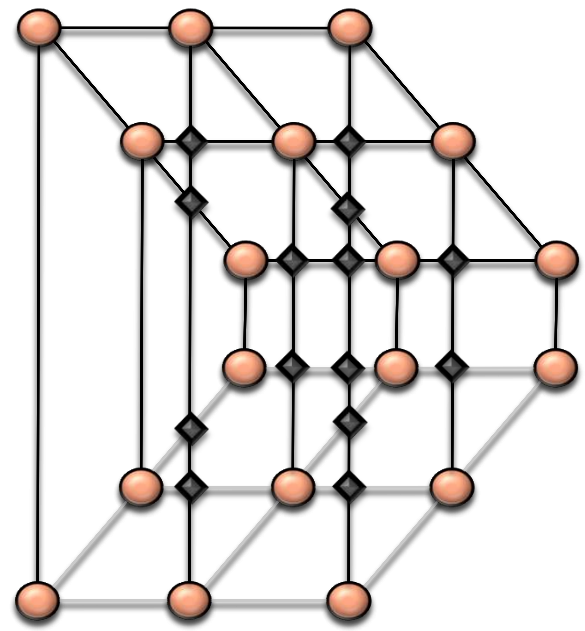
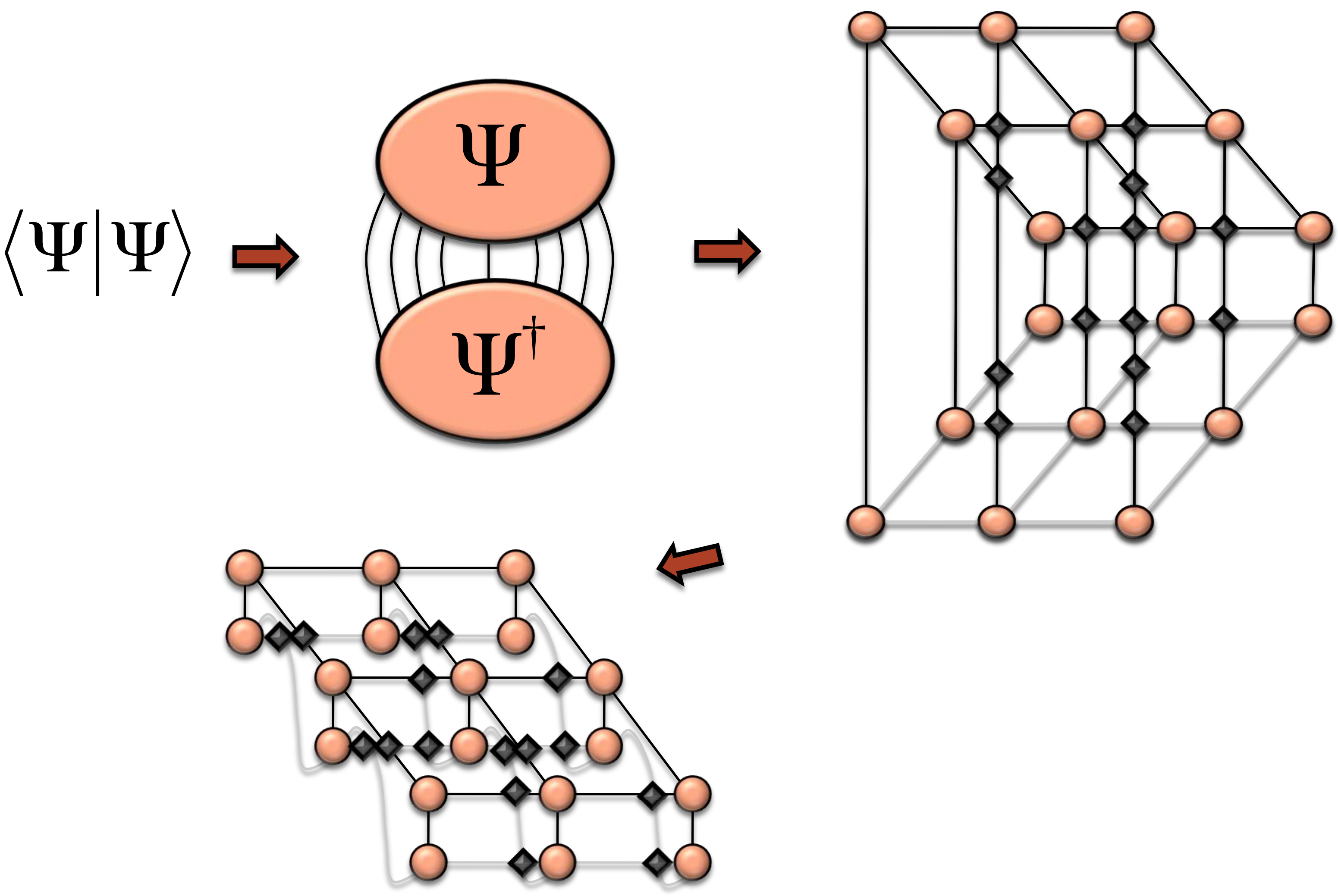


$\Rightarrow$



Example: *scalar product of 3x3 PEPS*



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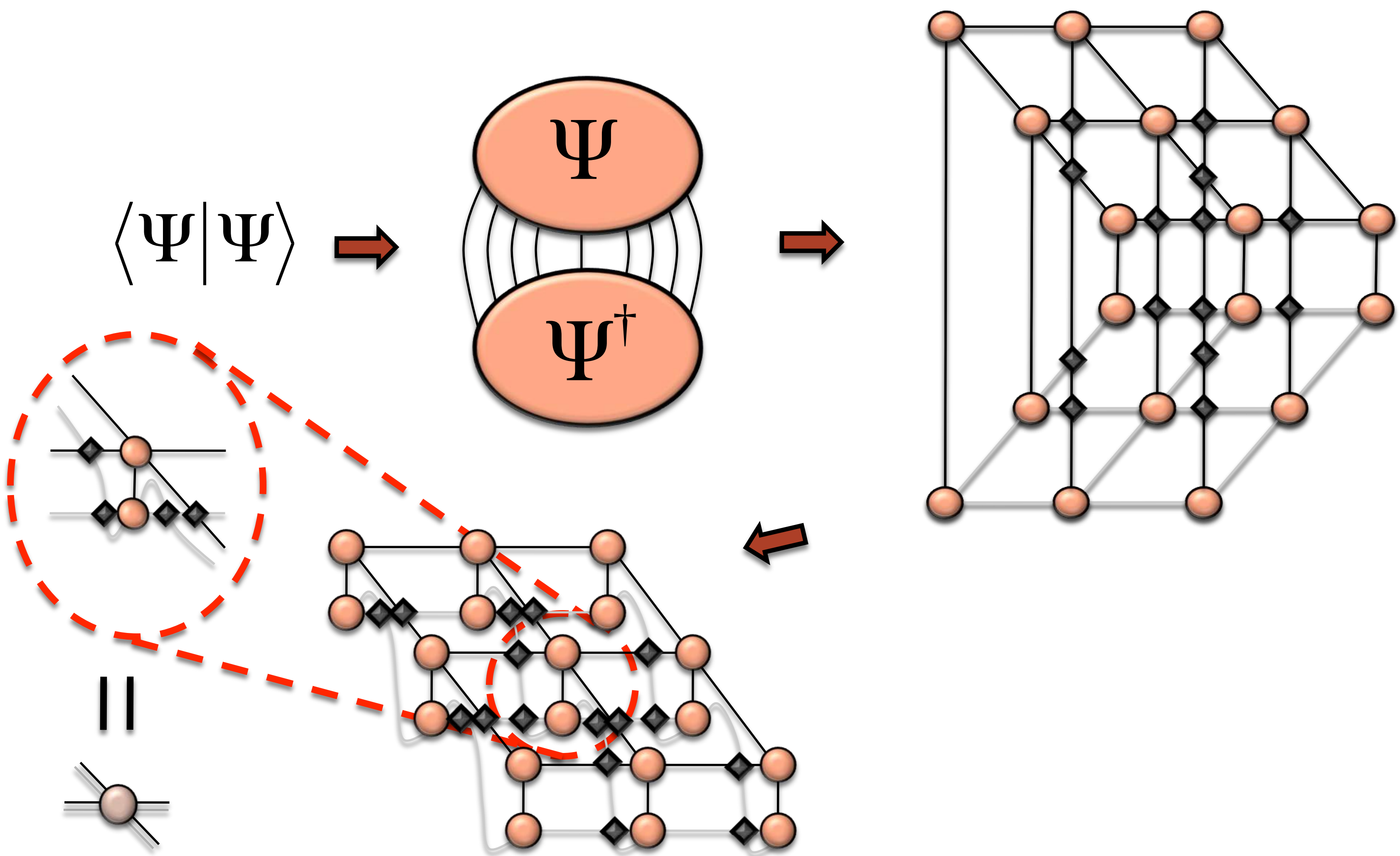

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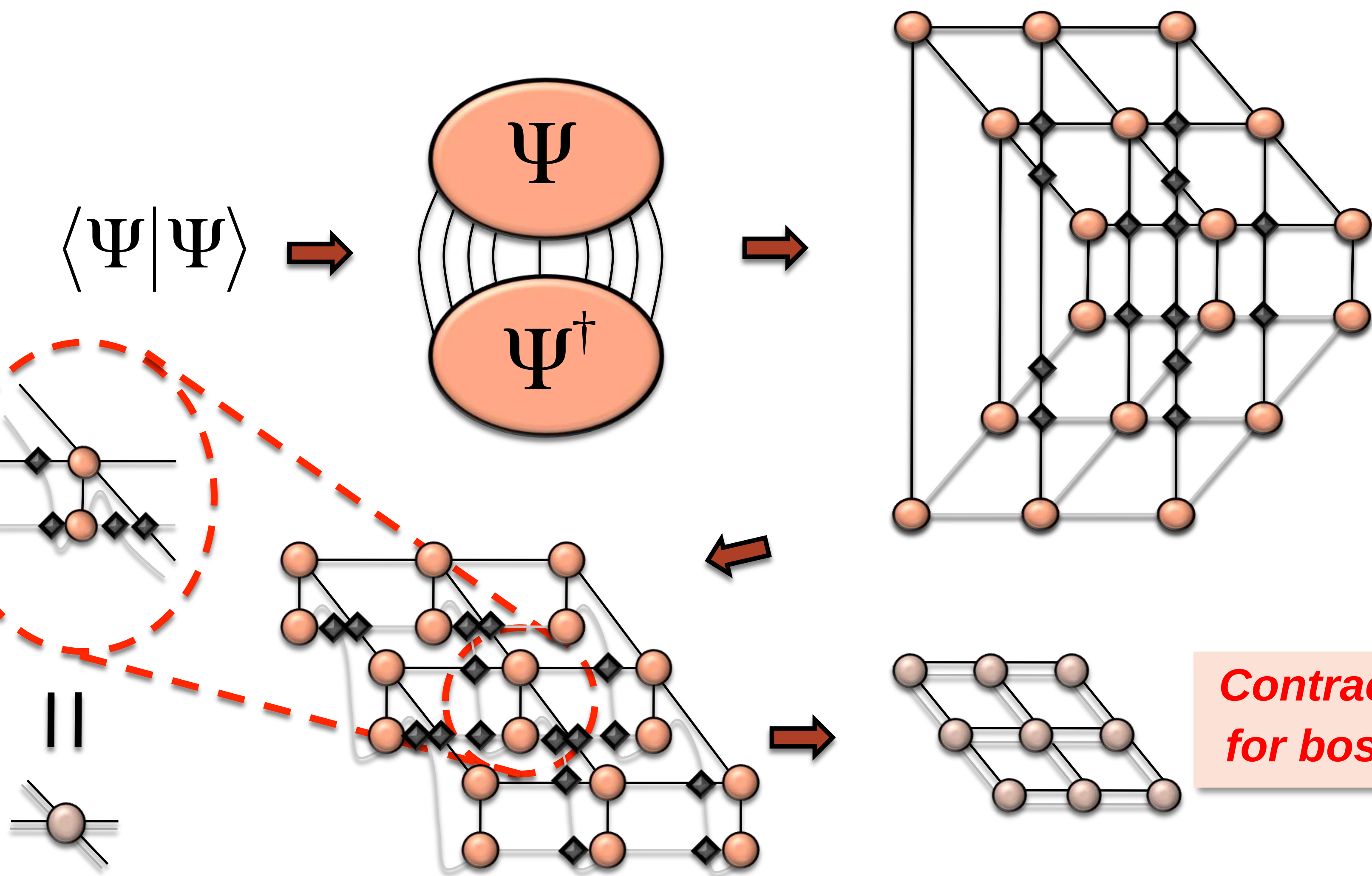

Contract as for bosons!

But... does it work?

But... does it work?

„Tensor networks provide today the best variational energies for the Hubbard model in the strong coupling limit. iPEPS has really made it“.

Matthias Troyer (at the Korrelationstage 2015)

*J. Jordan, RO, G. Vidal, F. Verstraete, I. Cirac, PRL **101** 250602 (2008)*
*P. Corboz, RO, B. Bauer, G. Vidal, PRB **81** 165104 (2010)*

YES, it does

*P. Corboz, PRB **93** 045116 (2016)*

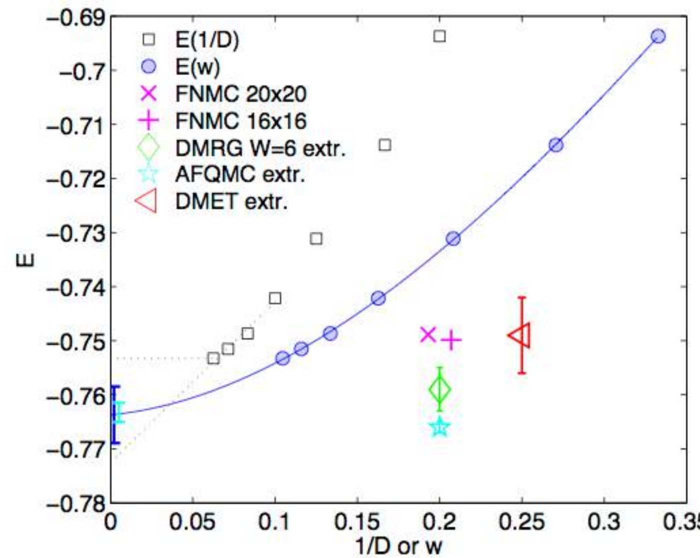

FIG. 4. (Color online) iPEPS energy of a period-5 stripe in the doped case in the strongly correlated regime ($U/t = 8$, $n = 0.875$) in comparison with other methods.

Outline

1) Entanglement and 1d MPS

2) 2d PEPS

2.1) Basics

2.2) Computing

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3) Some further topics

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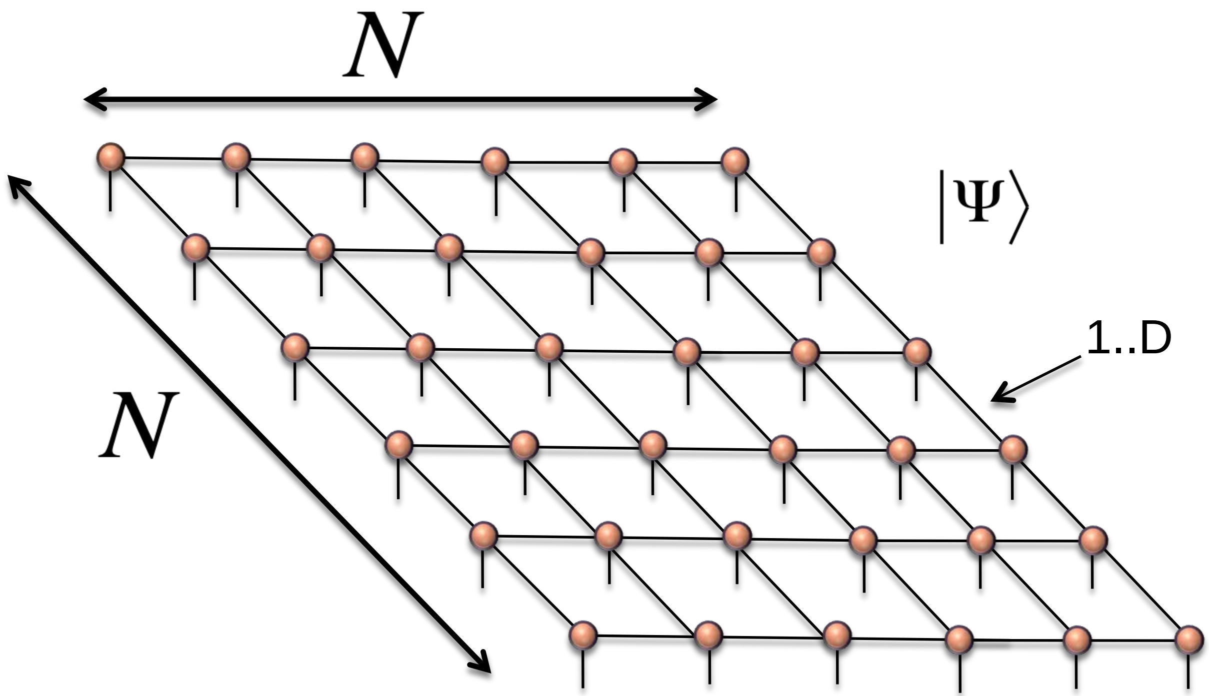
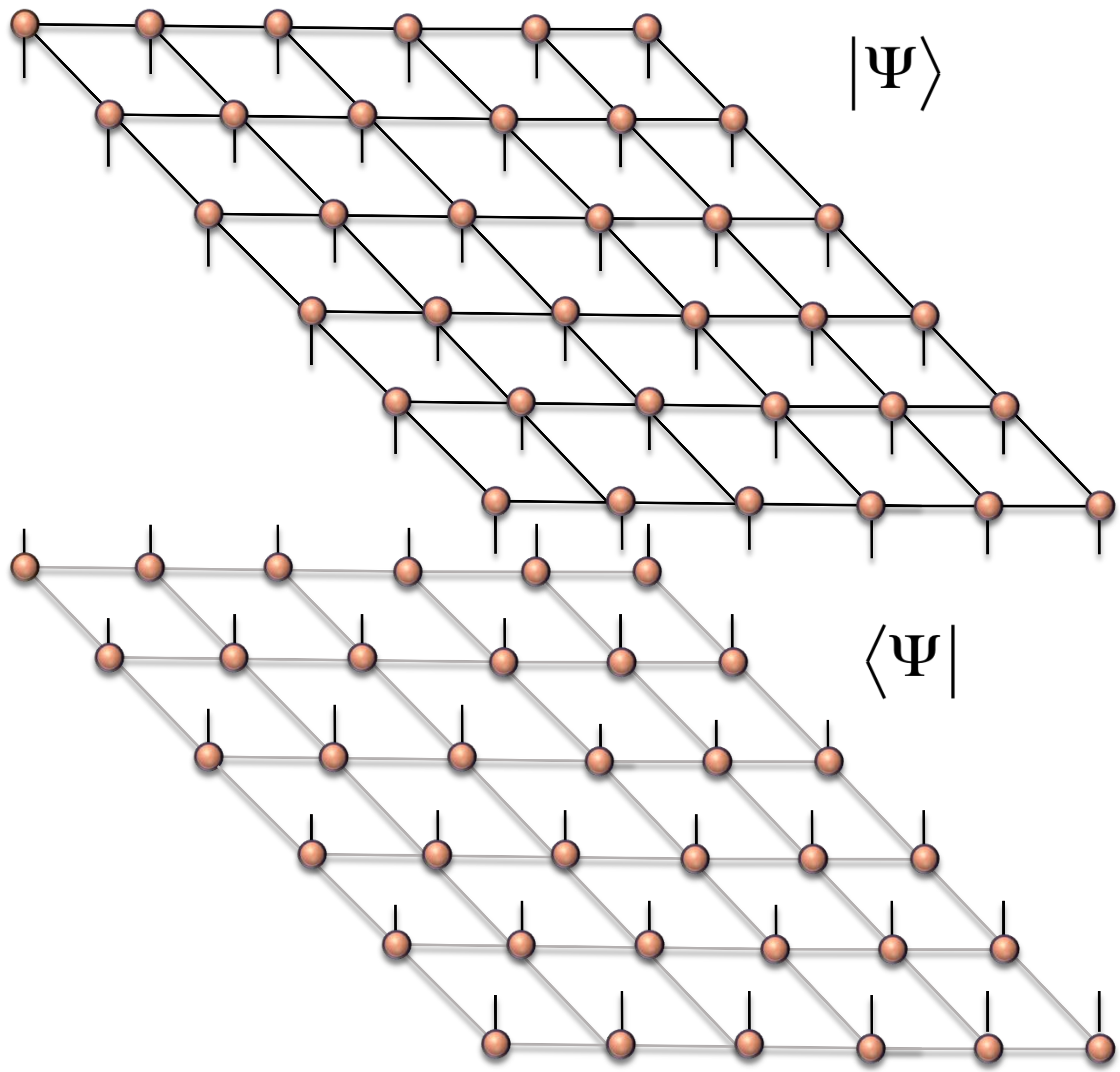
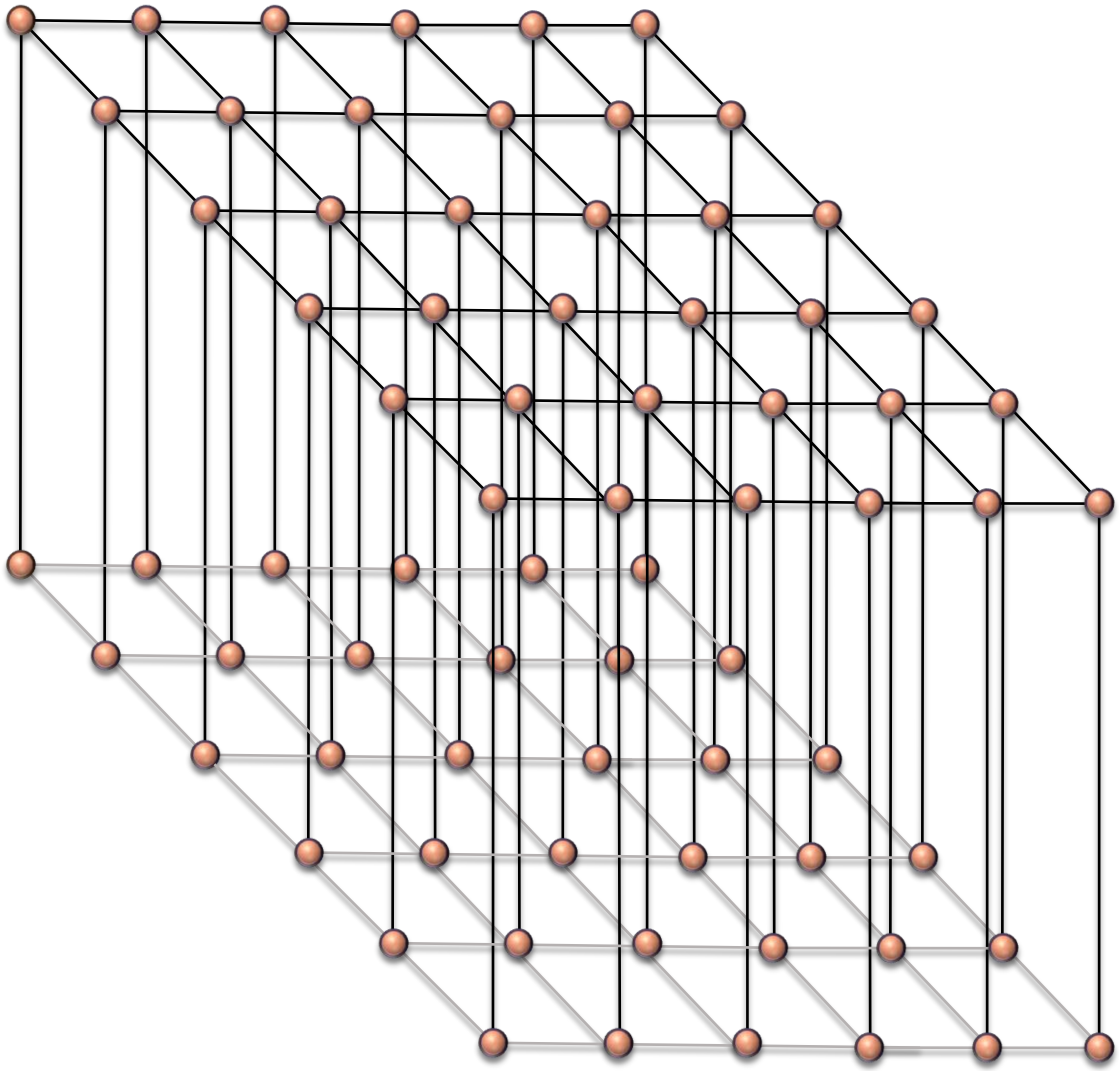
PEPS & Entanglement Hamiltonians

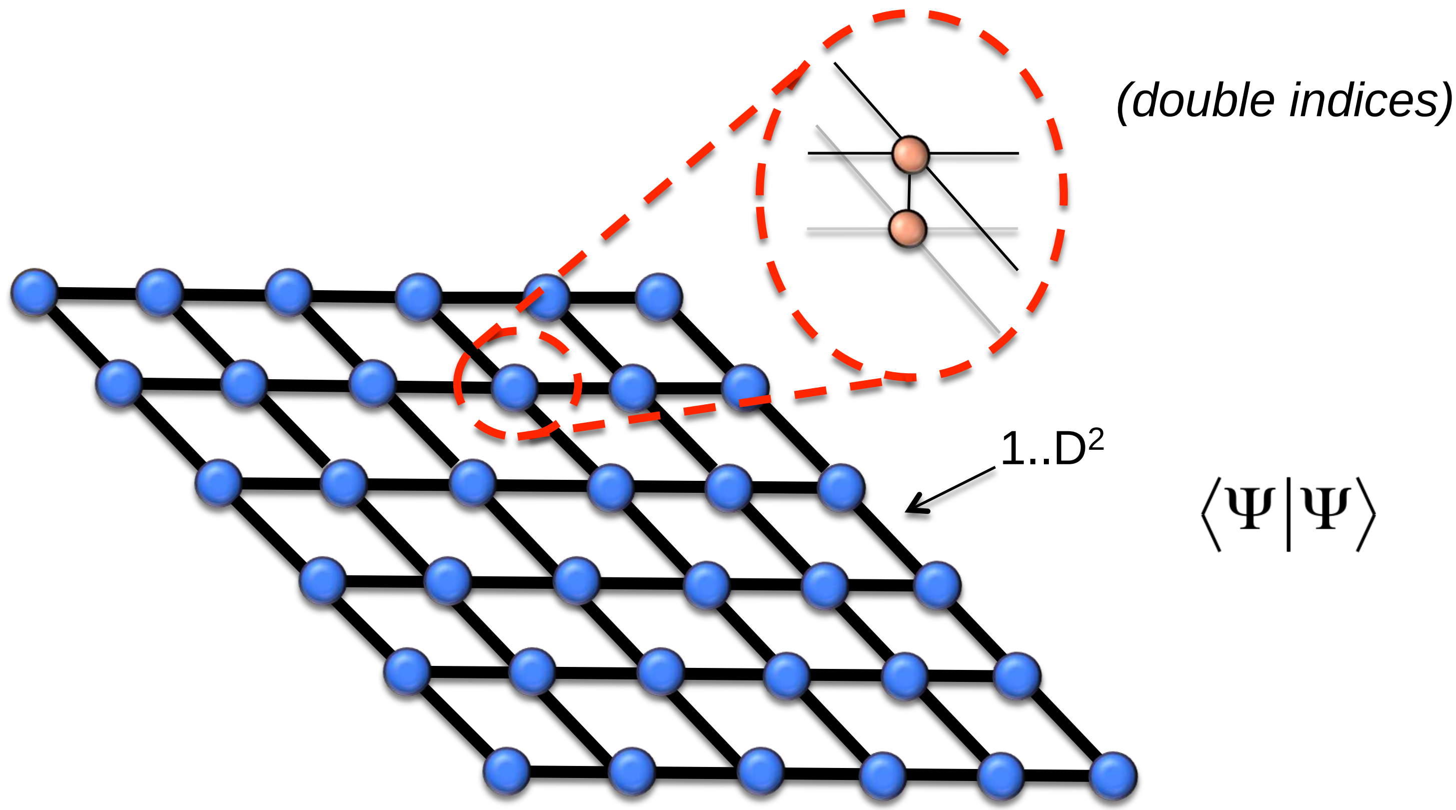
e.g. I. Cirac et al, PRB 83, 245134 (2011), N. Schuch et al, PRL 111, 090501 (2013)

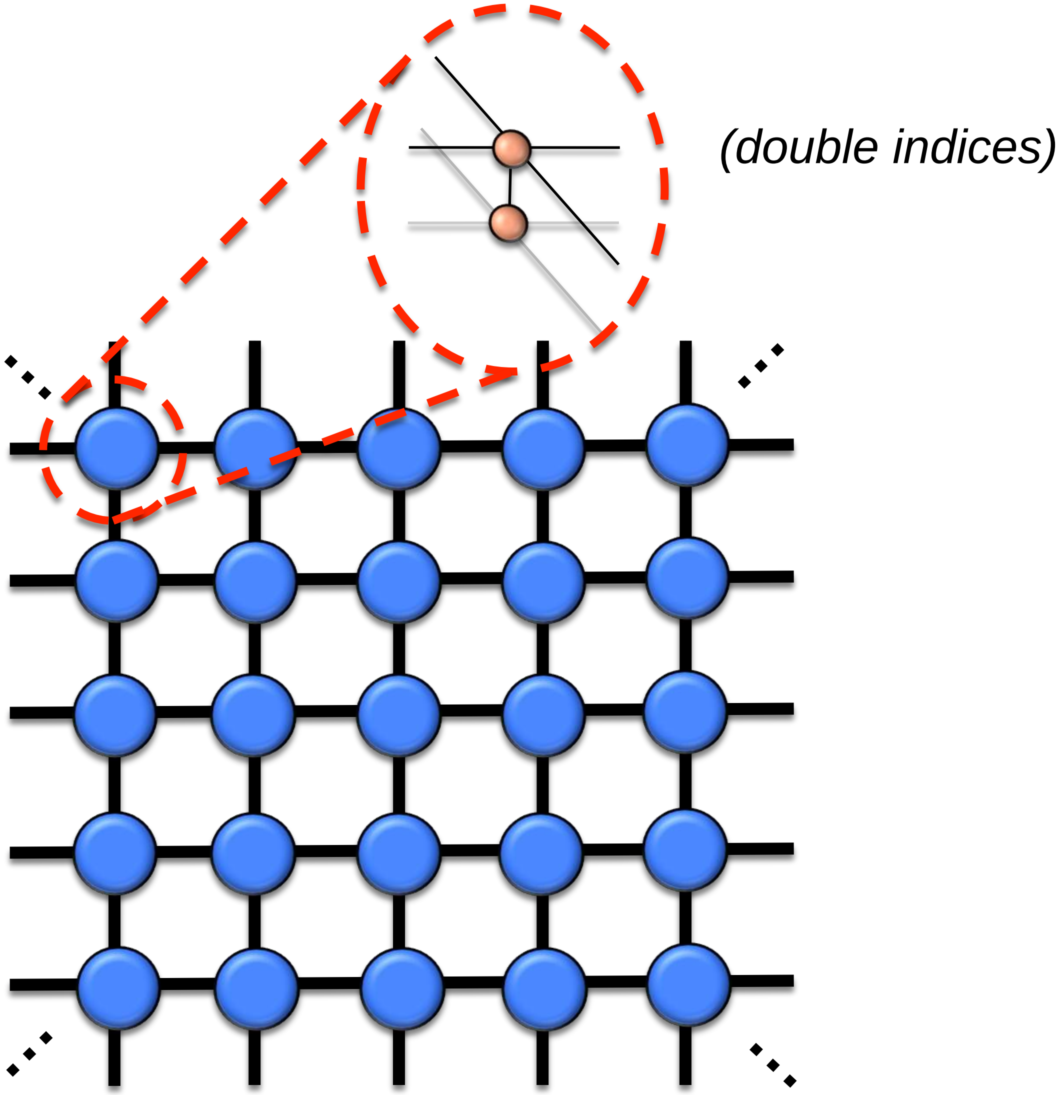

N
N
|Ψ⟩
1..D

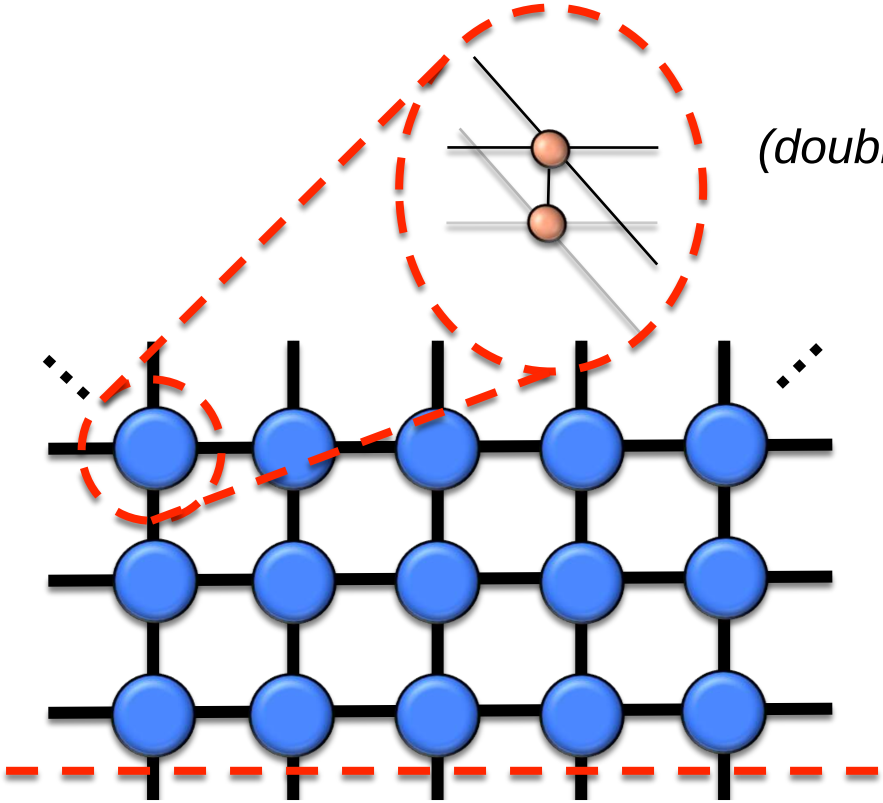

$|\Psi\rangle$
$\langle\Psi|$



$$\langle \Psi | \Psi \rangle$$


(double indices)
1..D²
$\langle \Psi | \Psi \rangle$

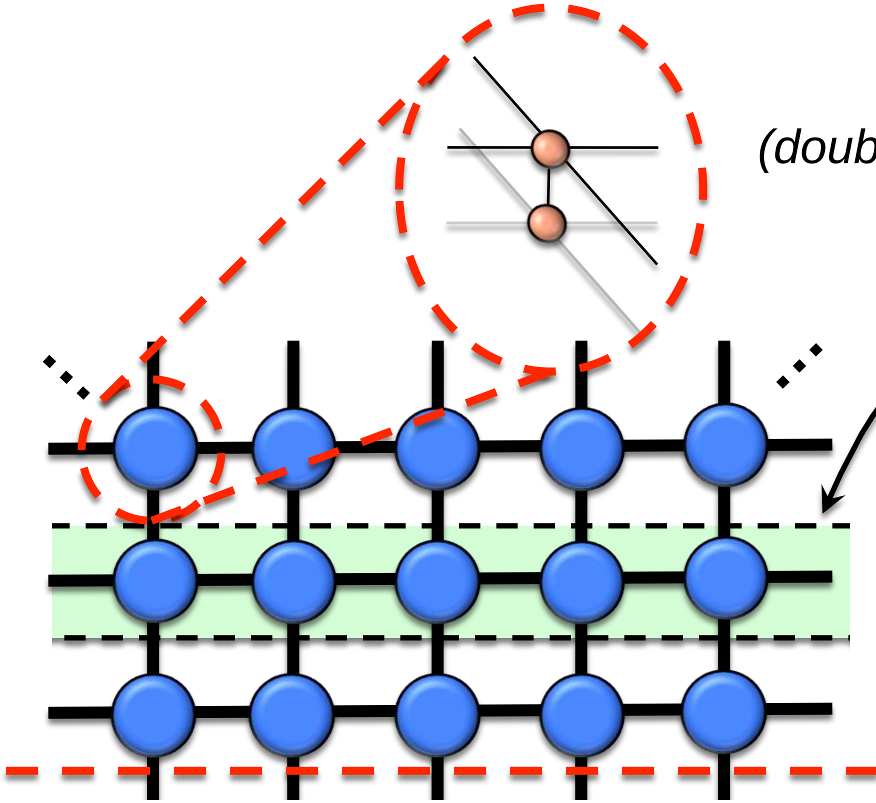

(double indices)



(double indices)

Boundary

How is physics described here?

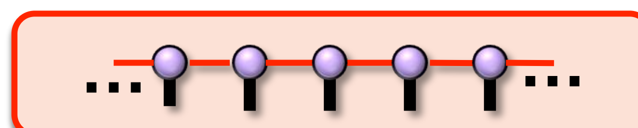


(double indices)

1-dim transfer matrix:

dominant eigenvector?

Can be approximated using infinite MPS

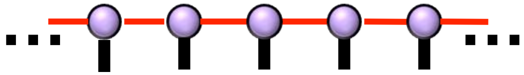


iTEBD, iDMRG, PWFRG, etc

Boundary

How is physics described here?

Emergent Hamiltonians

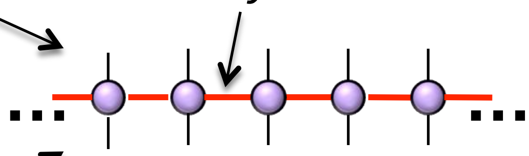


Remember it has double indices…

Emergent Hamiltonians

Virtual indices of bra
1…D

Boundary virtual index 1... χ



Virtual indices of ket
1…D

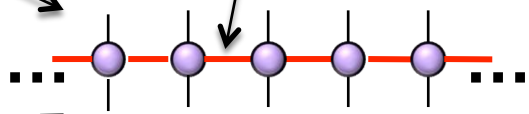
It is also hermitian and positive by construction (up to finite- χ effects)

Emergent Hamiltonians

Virtual indices of bra 1…D

Boundary virtual index 1... χ

Virtual indices of ket 1…D



It is also hermitian and positive by construction (up to finite- χ effects)

||

1d Entanglement Hamiltonian

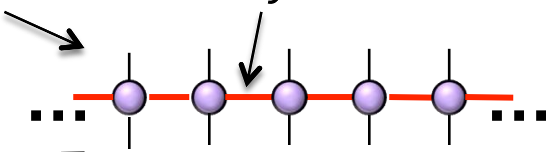
$$\rho = \exp(-H_E)$$

Who is H_E ???

Emergent Hamiltonians

Virtual indices of bra 1…D

Boundary virtual index 1… χ



Virtual indices of ket 1…D

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||

1d Entanglement Hamiltonian

$$\rho = \exp(-H_E)$$

Who is H_E ???

Bulk	*Correspondence*	*Boundary*
Gapped 2d systems, trivial phase	→	*1d Hamiltonian, short-range*
Critical 2d systems	→	*1d Hamiltonian, long-range*
Gapped 2d systems, topological order	→	*Completely non-local (projector)*
Chiral topological order, gapless	→	*(1+1)d Conformal field theory*

RO, M. Mambrini, D. Poilblanc, work in progress

Particles and energies from Hamiltonians, and Hamiltonians from networks of entanglement + bulk-boundary correspondence

TNs with symmetries

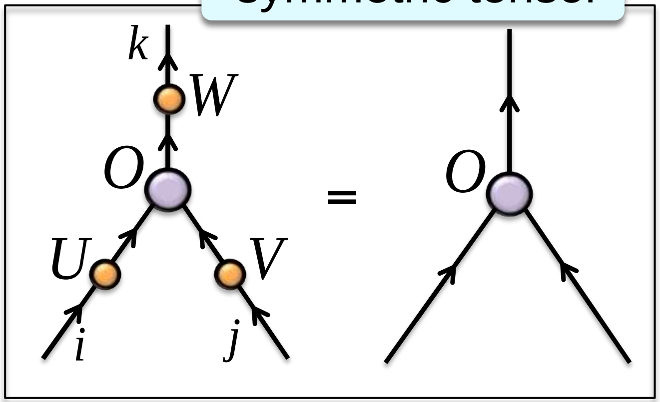
e.g., S. Singh, R. N. C. Pfeifer, G. Vidal, PRA 82, 050301 (2010)

c.f. M. Oshikawa's talk

Symmetric tensors and Schur's lemma

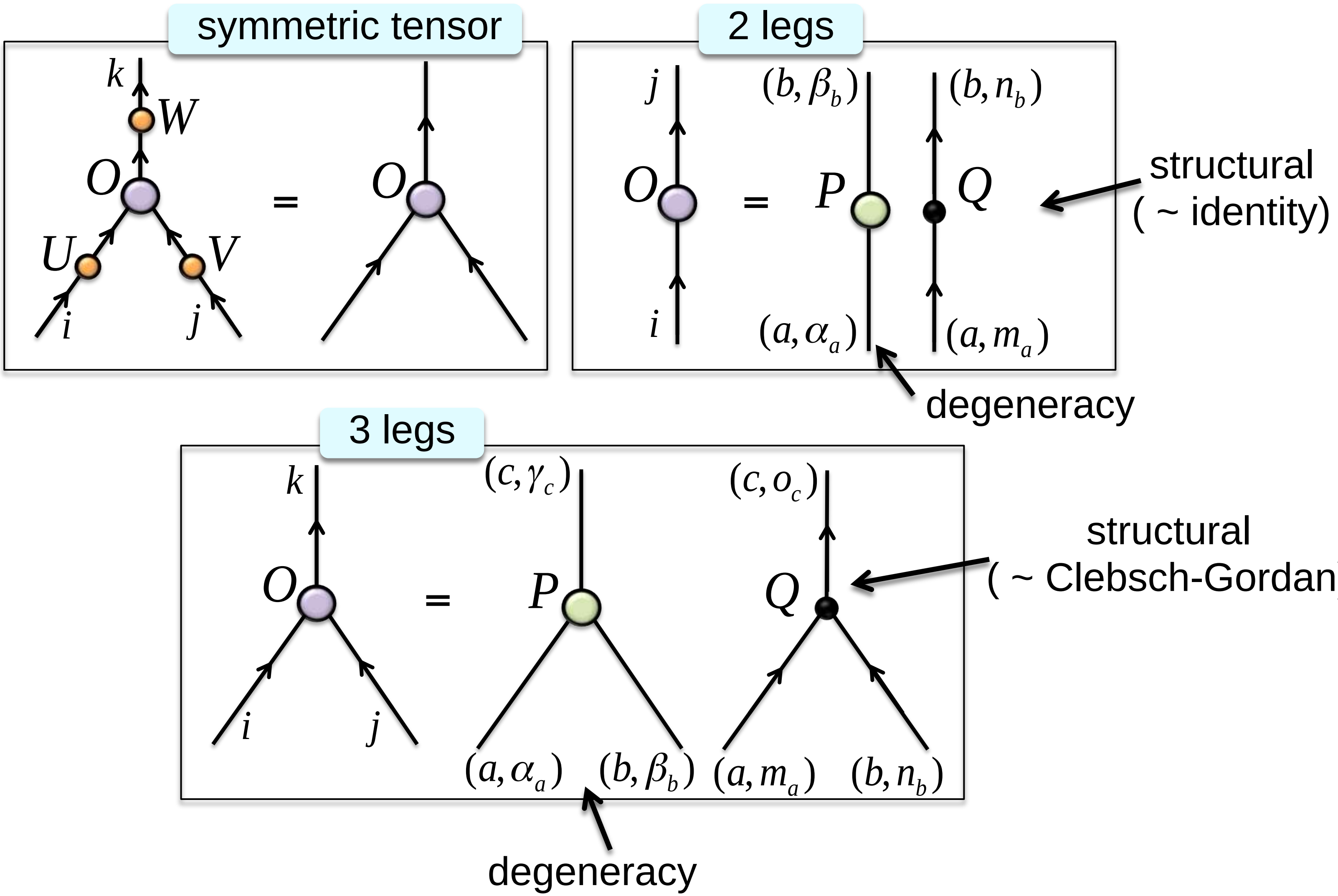
e.g., S. Singh, R. N. C. Pfeifer, G. Vidal, PRA 82, 050301 (2010)

symmetric tensor



Symmetric tensors and Schur's lemma

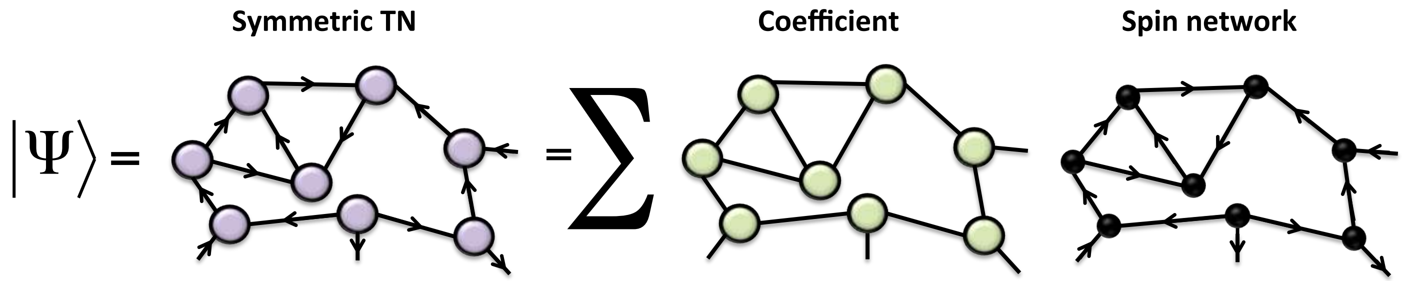
e.g., S. Singh, R. N. C. Pfeifer, G. Vidal, PRA 82, 050301 (2010)



Structural part depends only on the group properties (intertwiners)

Emergent spin networks

e.g., S. Singh, R. N. C. Pfeifer, G. Vidal, PRA 82, 050301 (2010)



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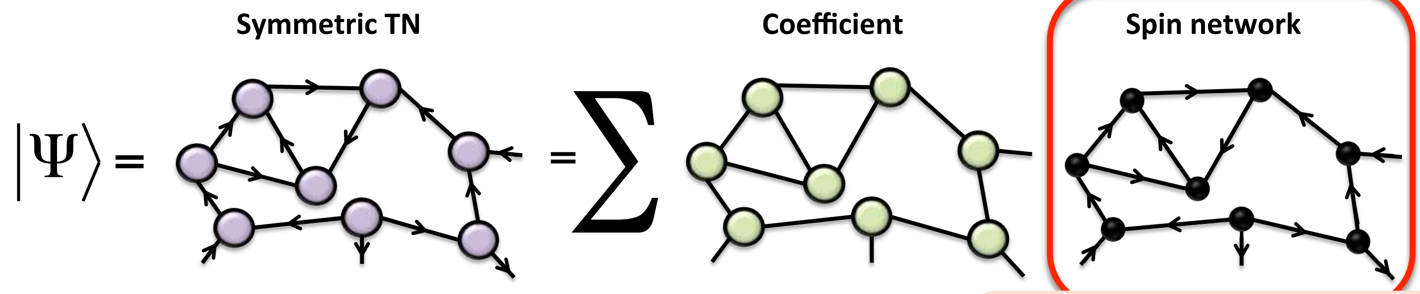

States of quantum geometry in loop quantum gravity...

Emergent spin networks

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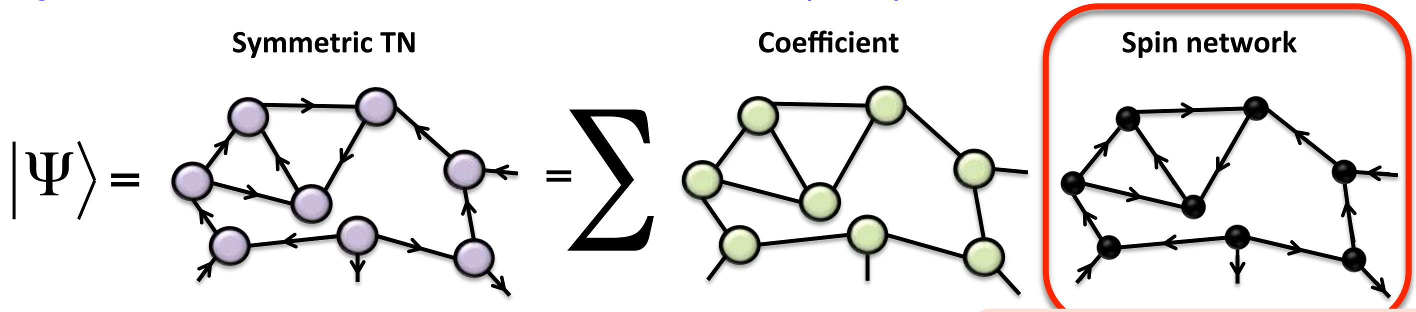


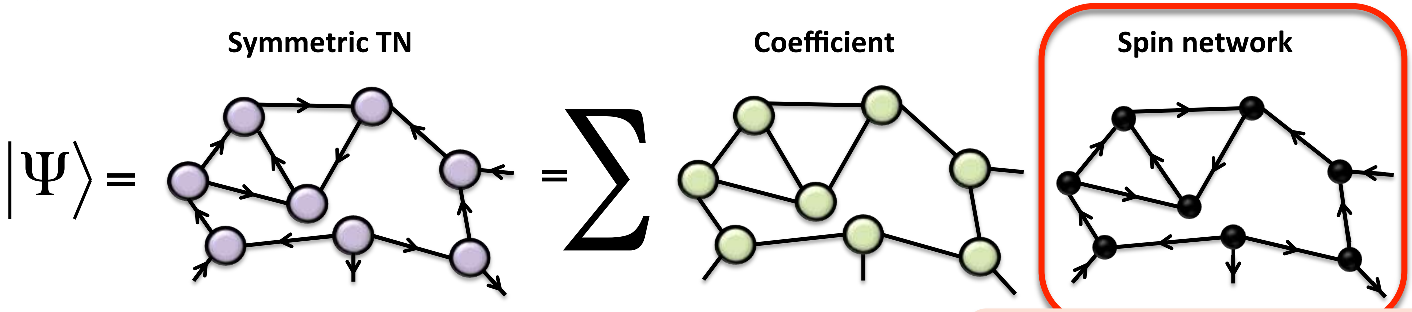
States of quantum geometry in loop quantum gravity...

And AdS/CFT and Ryu-Takayanagi and string theory and ER=EPR and cMERA and spacetime from entanglement and metrics from fidelities and wormholes and blackholes and firewalls and and and...

c.f. T. Takayanagi's talk

Emergent spin networks

e.g., S. Singh, R. N. C. Pfeifer, G. Vidal, PRA 82, 050301 (2010)



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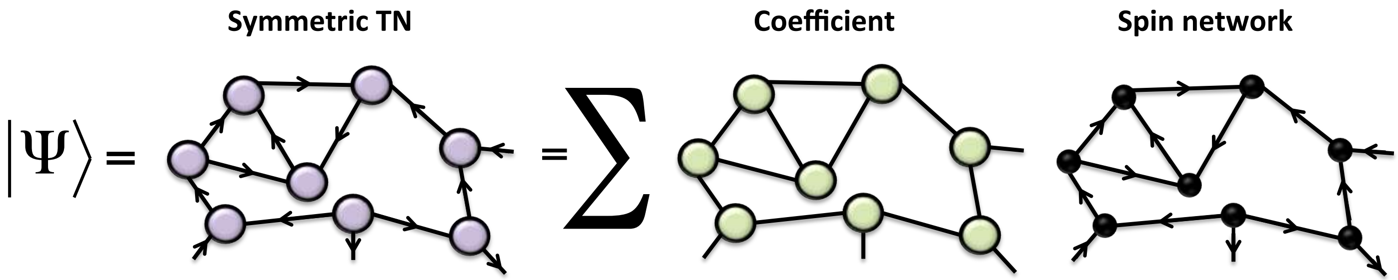

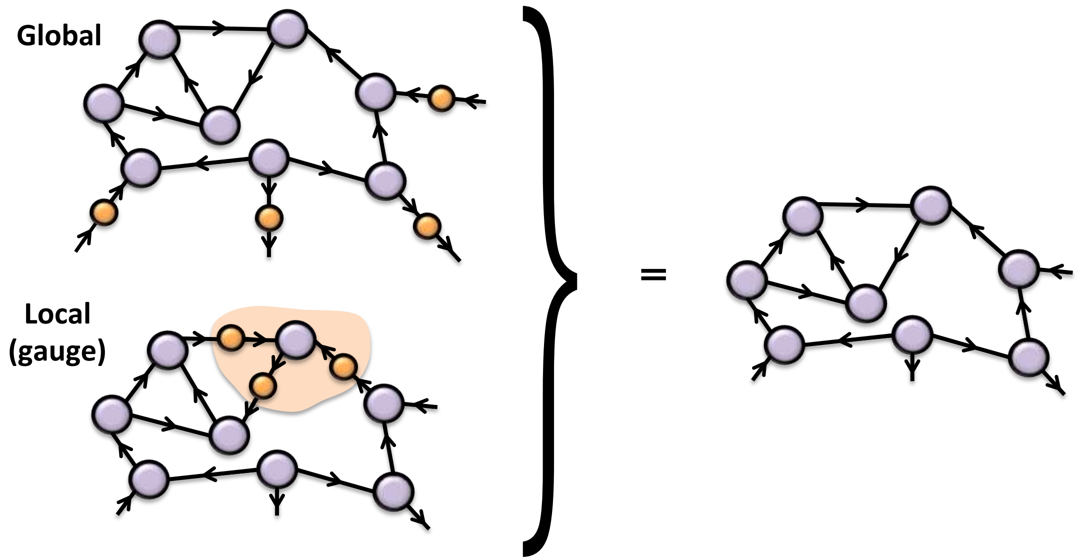

Global and gauge symmetries are handled naturally

Concerning numerics: HUGE computational savings, e.g., SU(2)-DMRG

Lots of other things:

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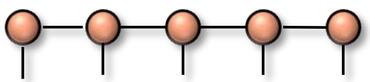
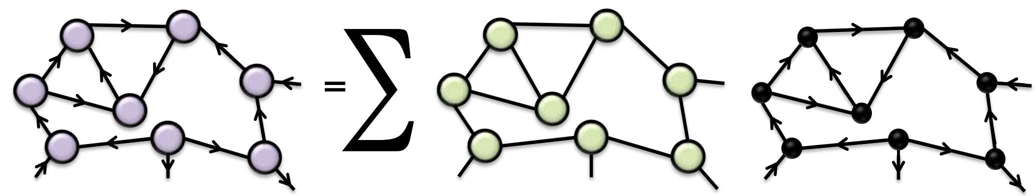
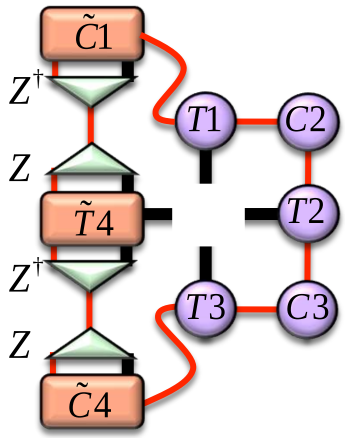
Chiral PEPS
Entanglement measures
Time evolution
Infinite systems
Classical systems
Corner Transfer Matrices
Tensor Renormalization Group
Variational updates
PESS
Phases of matter
3d PEPS
Topological systems
Excited states
Thermal states & dissipation
Continuous tensor networks
Tensor Networks & Montecarlo
Practical implementations
Blablabla...

Ask me
if interested!







Thank you!