Logarithmic Minimal Models, Critical Dense Polymers, Percolation and $\mathcal{W}$-Extended Fusion Rules

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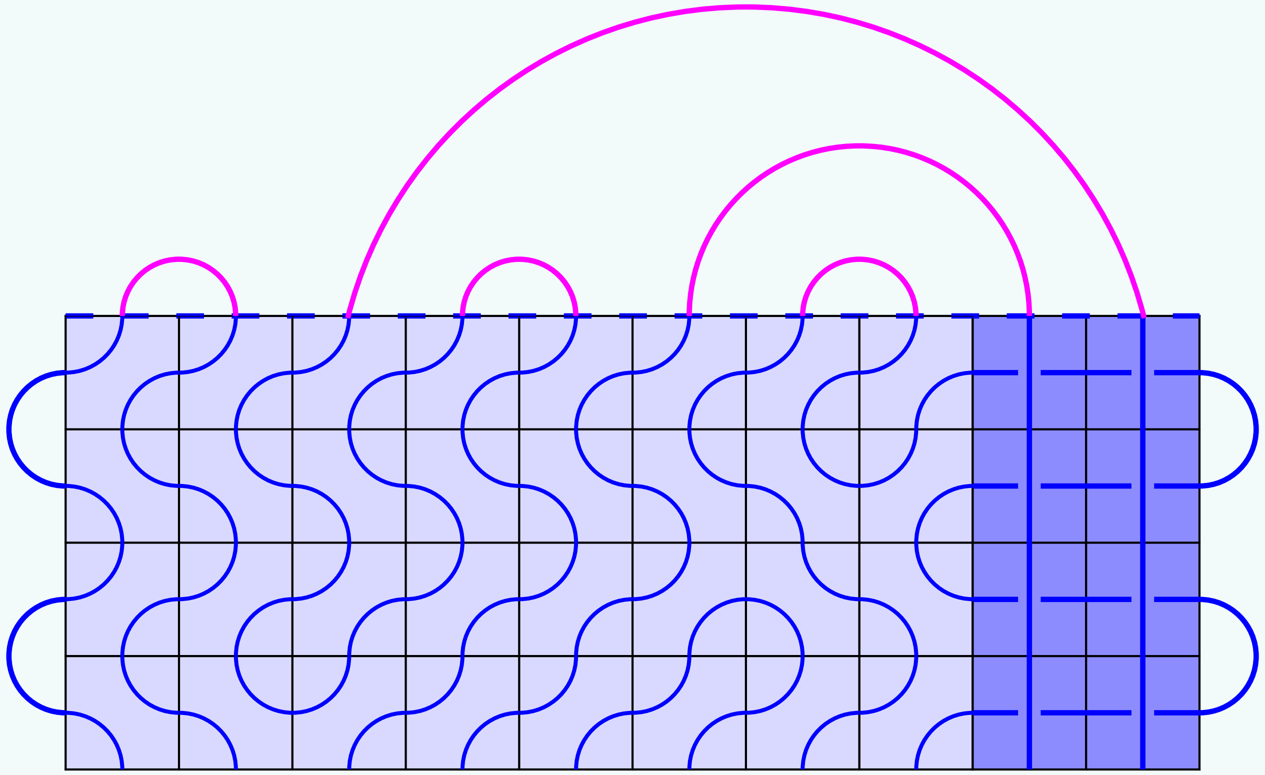

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- PAP, J.Rasmussen, *Solvable critical dense polymers*, J.Stat.Mech. P02015 (2007)
- J.Rasmussen, PAP, *Fusion algebras of logarithmic minimal models*, J.Phys. A40 13711–33 (2007)
- PAP, J.Rasmussen, P.Ruelle, *Integrable boundary conditions and $\mathcal{W}$-extended fusion of the logarithmic minimal models $\mathcal{LM}(1,p)$*, arXiv:0803.0785, J. Phys. A (2008)
- PAP, J.Rasmussen, *$\mathcal{W}$-extended fusion of critical percolation*, arXiv:0804.4335, J. Phys. A (2008)
- J.Rasmussen, *$\mathcal{W}$-extended logarithmic minimal models*, arXiv:0805.2991 (2008)

Some Background

1957– Broadbent & Hammersley: Percolation

1972– de Gennes, des Cloizeaux: Polymers

1986– Saleur, Duplantier: Conformal theory of polymers, percolation

1993– Gurarie: Logarithmic operators in CFT

1995– Kausch: Symplectic fermions

1992– Rozansky, Read, Saleur, Schomerus, etc: Supergroup Approach to Log CFT

1996– Rohsiepe, Flohr, Gaberdiel, Kausch, Feigin et al: Algebraic Approach to Log CFT

2006– Pearce, Rasmussen & Zuber: Lattice Approach to Log CFT

Lattice Approach: For Potts, RSOS models, . . .

local degrees of freedom	$\Rightarrow$	symmetric transfer matrices	$\Rightarrow$	diagonalizable transfer matrices	$\Rightarrow$	no rank ≥ 2 indecomposable representations	$\Rightarrow$	not logarithmic theory

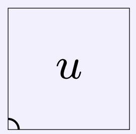
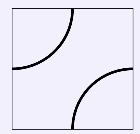
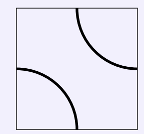
Paradigm Shift:

- Statistical systems with local "point" degrees of freedom yield rational CFTs.
- Polymers and percolation do not have any local degrees of freedom only nonlocal "string" degrees of freedom (polymers, connectivities) and are associated with Logarithmic CFTs . . .

logarithmic theory	$\Rightarrow$	nonlocal degrees of freedom

Logarithmic Minimal Models $\mathcal{LM}(p,p')$

- **Face operators defined in planar Temperley-Lieb algebra (Jones 1999)**

$X(u) =$  $= \sin(\lambda-u)$  $+ \sin u$  ; $\qquad X_j(u) = \sin(\lambda-u)\, I + \sin u\, e_j$

$1 \le p < p'$ coprime integers, $\qquad \lambda = \dfrac{(p'-p)\pi}{p'} =$ crossing parameter

$u =$ spectral parameter, $\qquad \beta = 2\cos\lambda =$ fugacity of loops (closed strings)

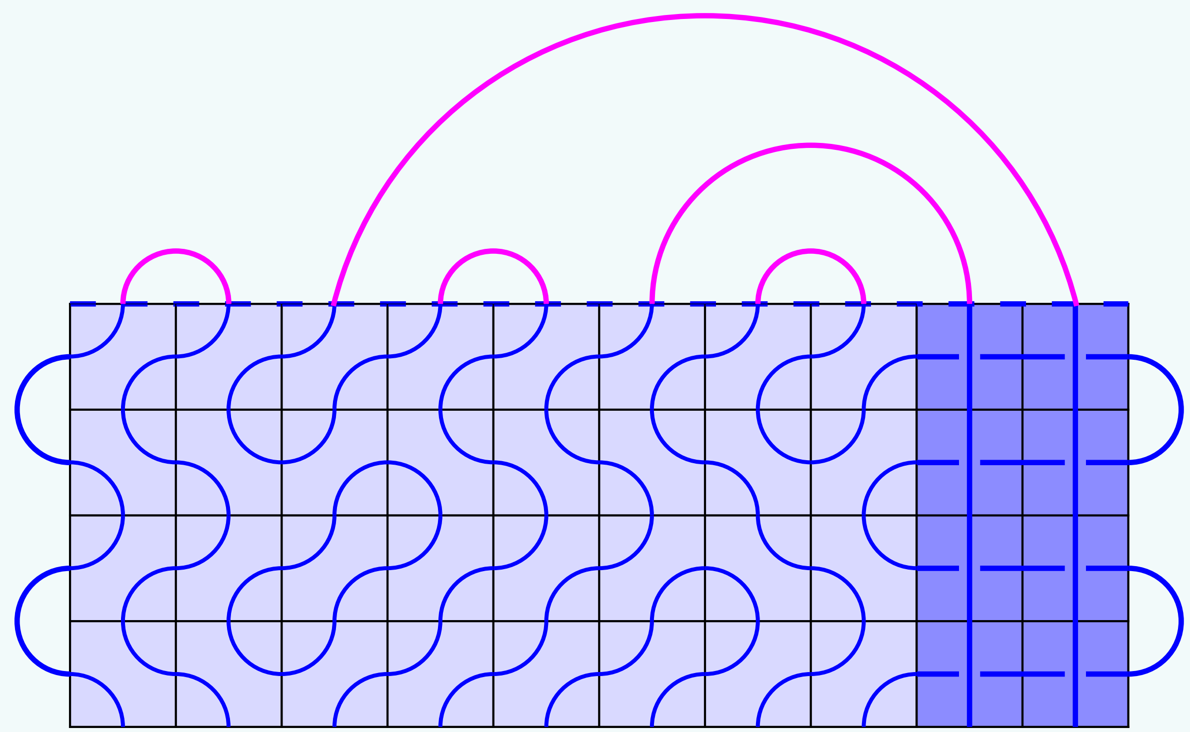
Planar Algebra
(Temperley-Lieb Algebra)

$\downarrow$ YBE

Nonlocal Statistical Mechanics
(Yang-Baxter Integrable Link Models)

continuum limit $\downarrow$ $\qquad \uparrow$ lattice realization

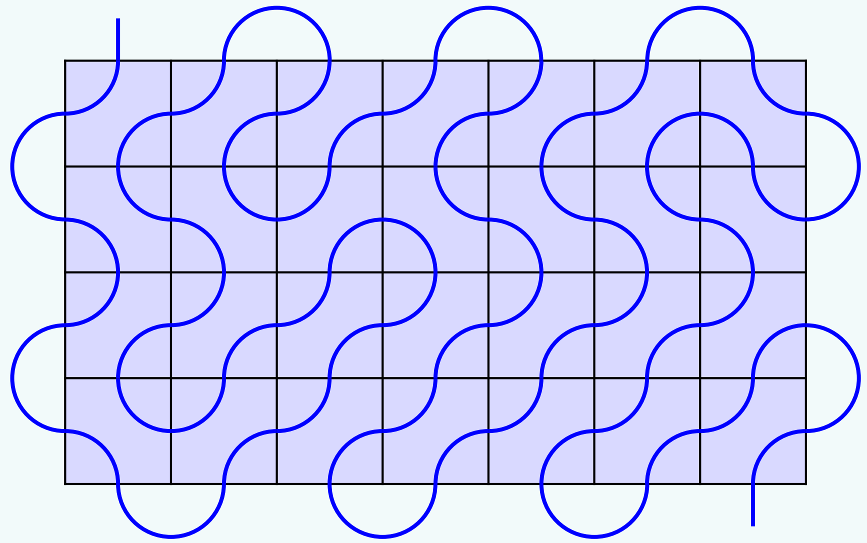
Logarithmic CFTs
(Logarithmic Minimal Models)



Nonlocal Degrees of Freedom = Strings

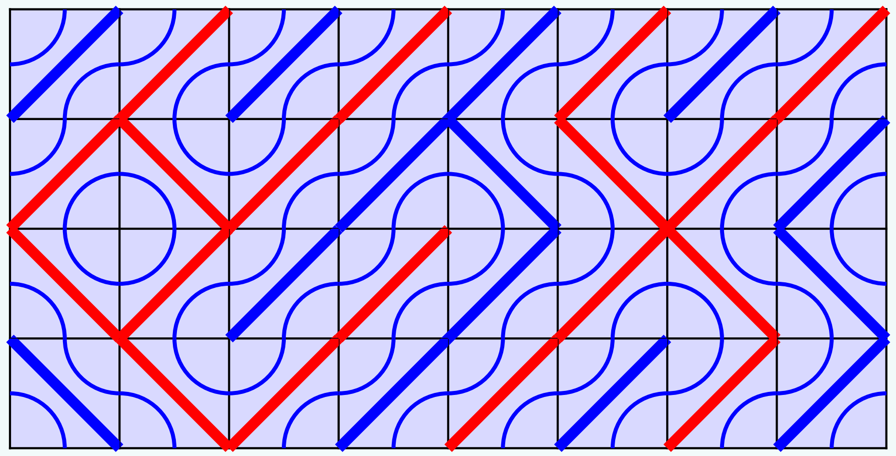
Polymers and Percolation on the Lattice

- **Critical Dense Polymers:** $(p,p')=(1,2)$, $\lambda=\frac{\pi}{2}$



$\beta=0 \;\Rightarrow\;$ no loops $\;\Rightarrow\;$ space filling dense polymer

- **Critical Percolation:** $(p,p')=(2,3)$, $\lambda=\frac{\pi}{3}$, $u=\frac{\lambda}{2}=\frac{\pi}{6}$ (isotropic)



Bond percolation on the blue square lattice:

Critical probability $=p_c=\sin(\lambda-u)=\sin u=\frac{1}{2}$

$\beta=1 \quad\Rightarrow\quad$ local stochastic process

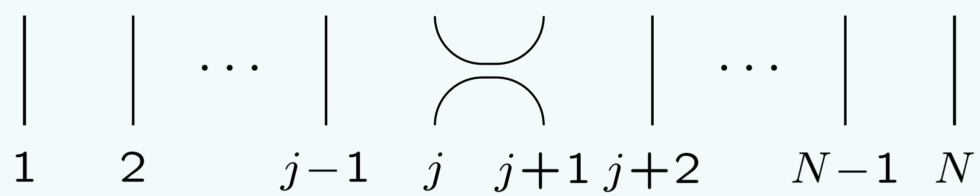
Linear Temperley-Lieb Algebra

- The linear TL algebra is generated by $e_1, \ldots, e_{N-1}$ and the identity I acting on N strings

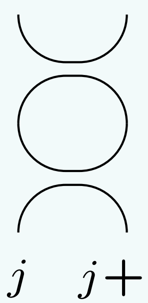
$$\boxed{\begin{cases} e_j^2 = \beta\, e_j, & \\ e_j\, e_k\, e_j = e_j, & |j-k| = 1, \qquad j,k = 1,2,\ldots,N-1; \qquad \beta = 2\cos\lambda \\ e_j\, e_k = e_k\, e_j, & |j-k| > 1 \end{cases}}$$

- The TL generators e_j are represented graphically by *monoids*

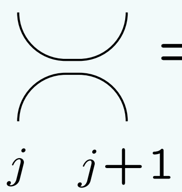
$e_j =$



$e_j^2 =$

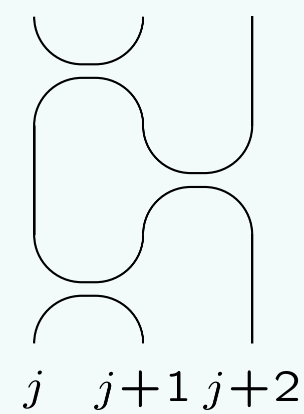


$= \beta$

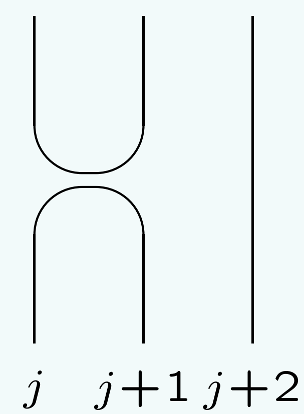


$= \beta\, e_j,$

$e_j e_{j+1} e_j =$



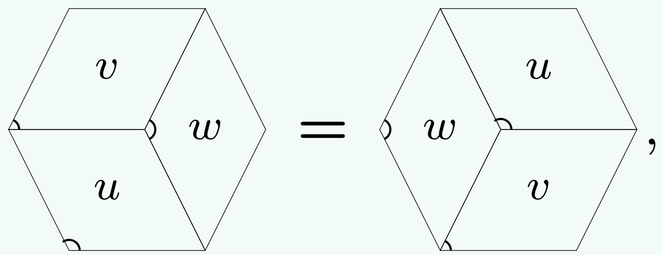
$=$



$= e_j$

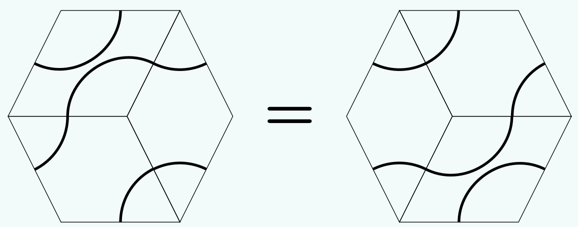
Integrability I: Yang-Baxter Equation (YBE)

- The YBE express the equality of two planar 3-tangles ($w = v - u$)

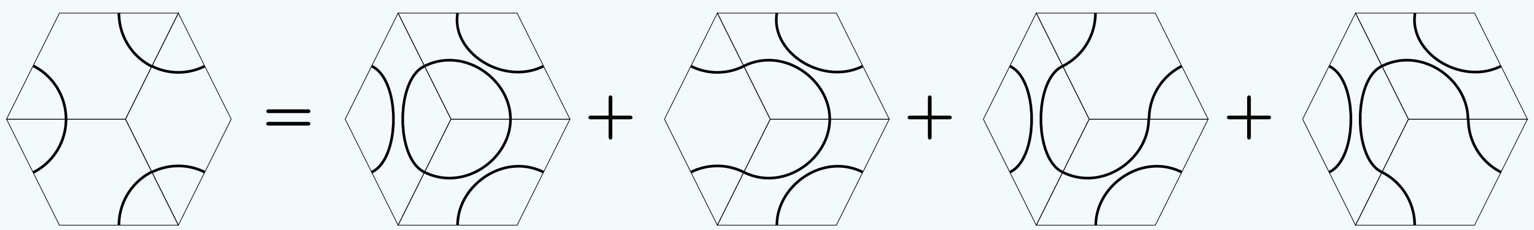


$$X_j(w)X_{j+1}(u)X_j(v) \;=\; X_{j+1}(v)X_j(u)X_{j+1}(w)$$

- The five possible connectivities of the external nodes give the diagrammatic equations



$\times\ 3\quad$ (120° rotations)



$\times\ 2\quad$ (180° rotations)

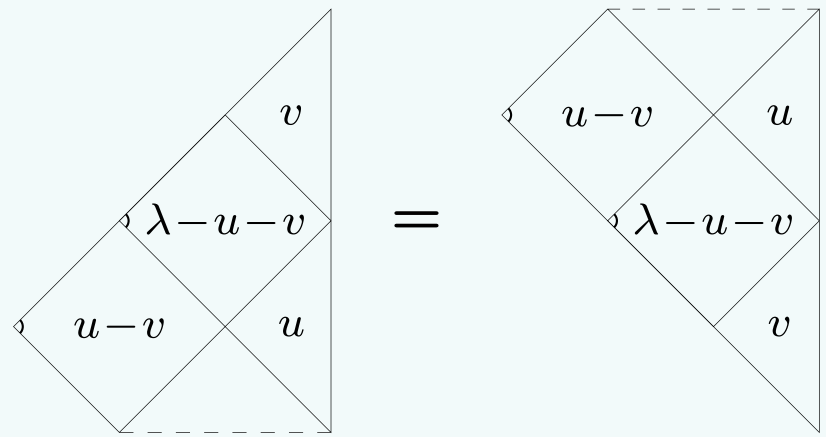
- The first equation is trivial. The second equation follows from the identity

$$\begin{aligned} s_1(-u)s_0(v)s_1(-w) \;&=\; \beta\, s_0(u)s_1(-v)s_0(w) \;+\; s_0(u)s_1(-v)s_1(-w) \\ &\quad+\; s_1(-u)s_1(-v)s_0(w) \;+\; s_0(u)s_0(v)s_0(w) \end{aligned}$$

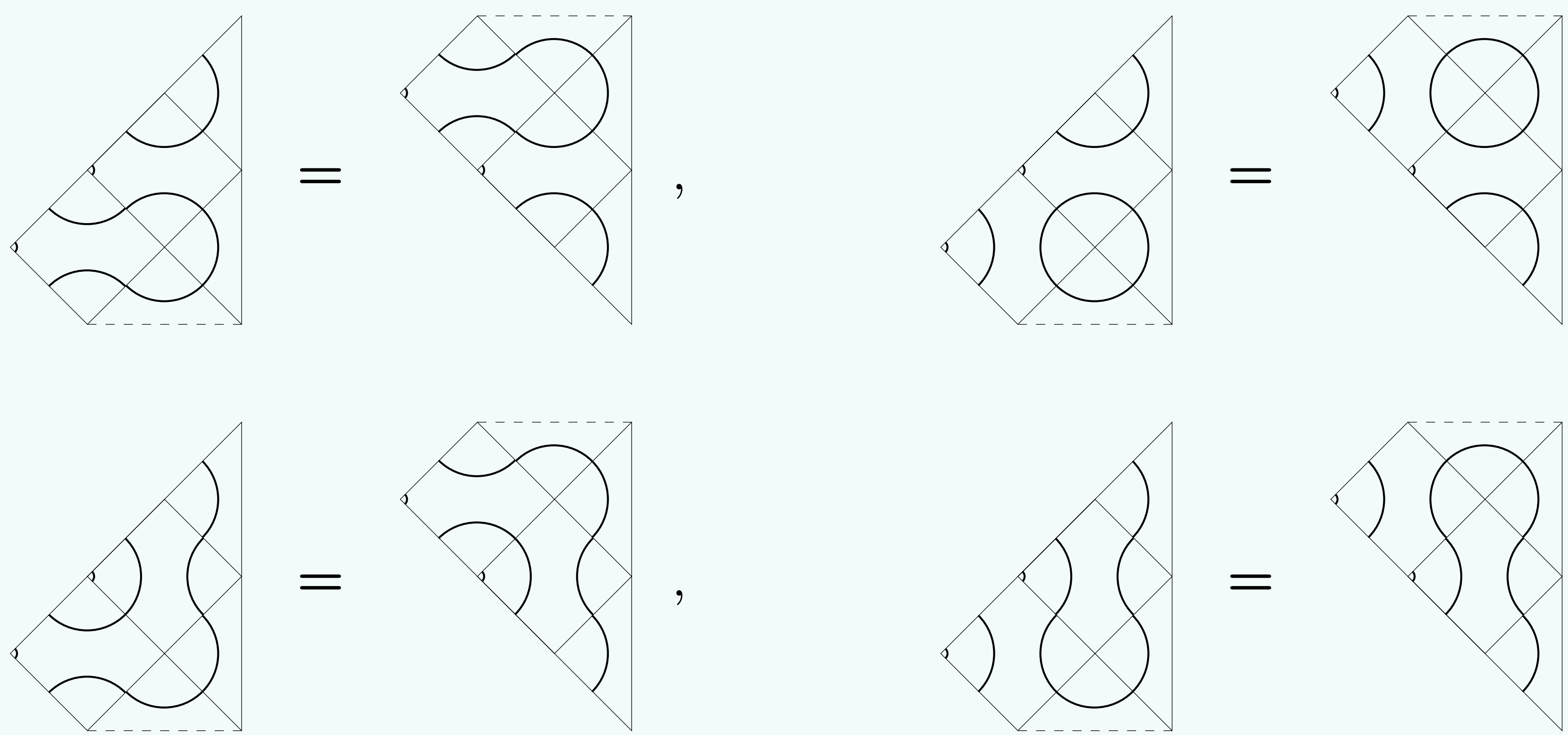
$$\boxed{s_r(u) = \frac{\sin(u + r\lambda)}{\sin\lambda}, \qquad \beta \;=\; 2\cos\lambda \;=\; \text{loop fugacity}}$$

Integrability II: Boundary Yang-Baxter Equation

- The Boundary Yang-Baxter Equation (BYBE) is the equality of boundary 2-tangles

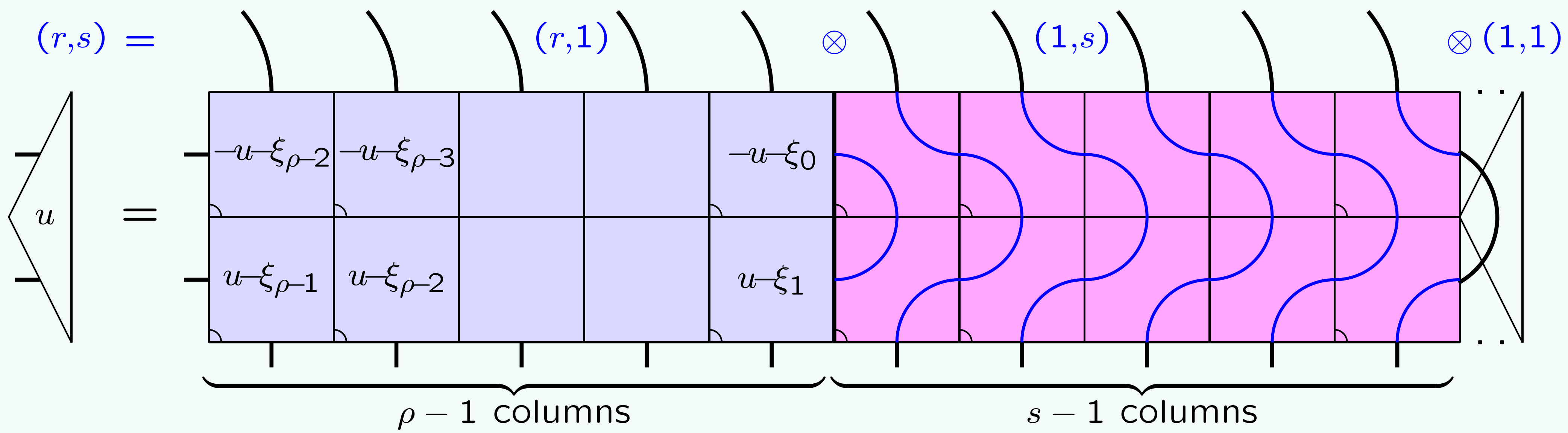


- For the elementary 1-triangle, this follows from four identities among the weights



Integrability III: Kac (r,s) BYBE Solutions

- For $r,s=1,2,3,\ldots$, the $(r,s)=(r,1)\otimes(1,s)$ BYBE solution is built as the fusion product of $(r,1)$ and $(1,s)$ integrable seams acting on the vacuum $(1,1)$ triangle:



- The column inhomogeneities are: $$\boxed{\xi_k=(k+k_0+\tfrac{1}{2})\lambda}$$

- There is at least one choice of the integers ρ and k_0 for each r.

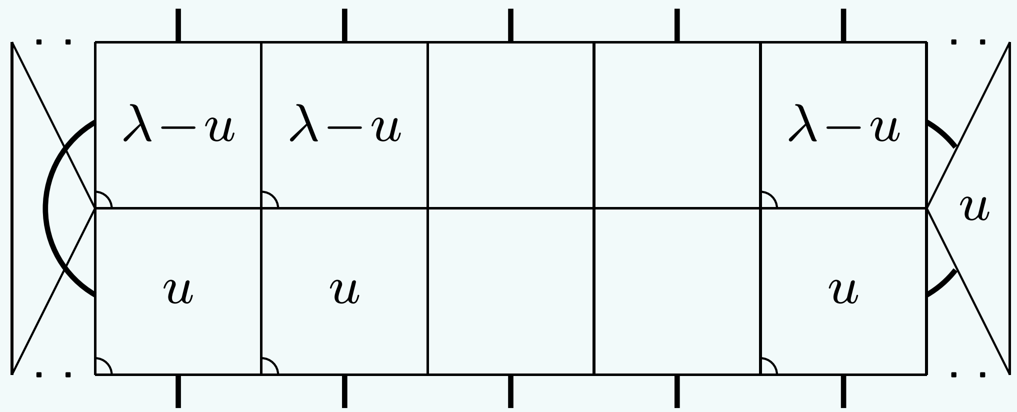
- The $\rho+s-2$ columns are considered part of the right boundary.
The arches at the top close to the left with up to $\rho+s-2$ defects propagating in the bulk.
The r-arches can not close among themselves and similarly for the s-arches.
But some of the s-arches can close with some of the r-arches.

- Left boundary solutions (r',s') are constructed similarly.

Double-Row Transfer Matrices

- For a strip with N columns, the double-row transfer "matrix" is the N-tangle

$D(u)$ =



- Using the Yang-Baxter (YBE) and Boundary Yang-Baxter Equations (BYBE) in the planar Temperley-Lieb (TL) algebra, it can be shown that, for any (r, s), these commute and are crossing symmetric

$$\boxed{D(u)D(v) = D(v)D(u), \qquad D(u) = D(\lambda - u)}$$

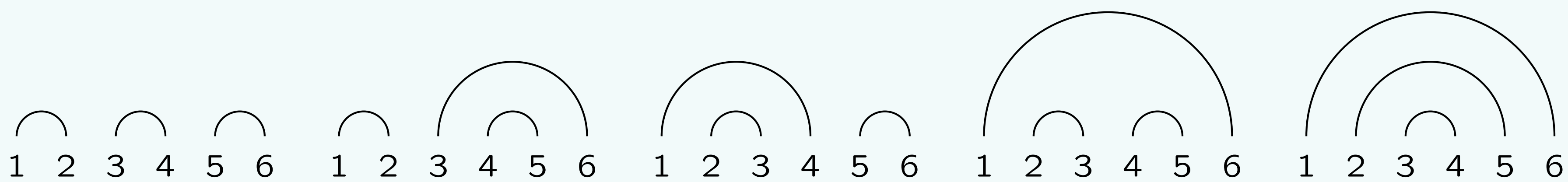
- Multiplication is vertical concatenation of diagrams, equality is the equality of N-tangles.

- In the case of one non-trivial boundary condition, the transfer matrices are found to be diagonalizable. For fusion, we take non-trivial boundary conditions on the left and right $(r', s') \otimes (r, s)$. In this case, the transfer matrices can exhibit Jordan cells and are not in general diagonalizable.

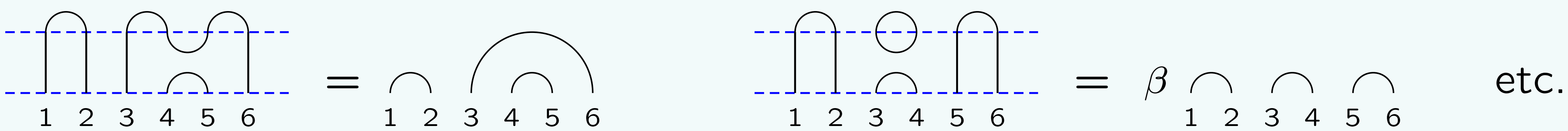
- The double row transfer matrices can be written in terms of the linear TL algebra. But it is necessary to act on a vector space of states to obtain *matrix representatives* of the operators in the planar algebra and their associated *spectra*.

Planar Link Diagrams

- The planar N-tangles act on a vector space $\mathcal{V}_N$ of *planar link diagrams*. The dimension of $\mathcal{V}_N$ is given by Catalan numbers. For $N = 6$, there is a basis of 5 link diagrams:



- The first link diagram is the reference state. Other states are generated by the action of the TL generators by concatenation from below



- The action of the TL generators on the states is nonlocal. It leads to matrices with entries $0, 1, \beta$ that represent the TL generators. For $N = 6$, the action of e_1 and e_2 on $\mathcal{V}_6$ is

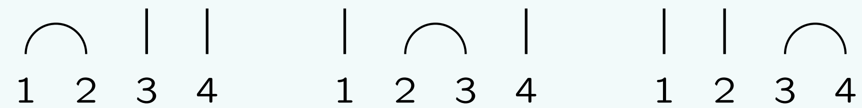
$$\boxed{e_1 = \begin{pmatrix} \beta & 0 & 1 & 0 & 1 \\ 0 & \beta & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & \beta & 0 & 0 \\ 0 & 1 & 0 & \beta & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \text{etc.}}$$

- Despite the symmetry of the monoid diagrams, they are not transpose symmetric.

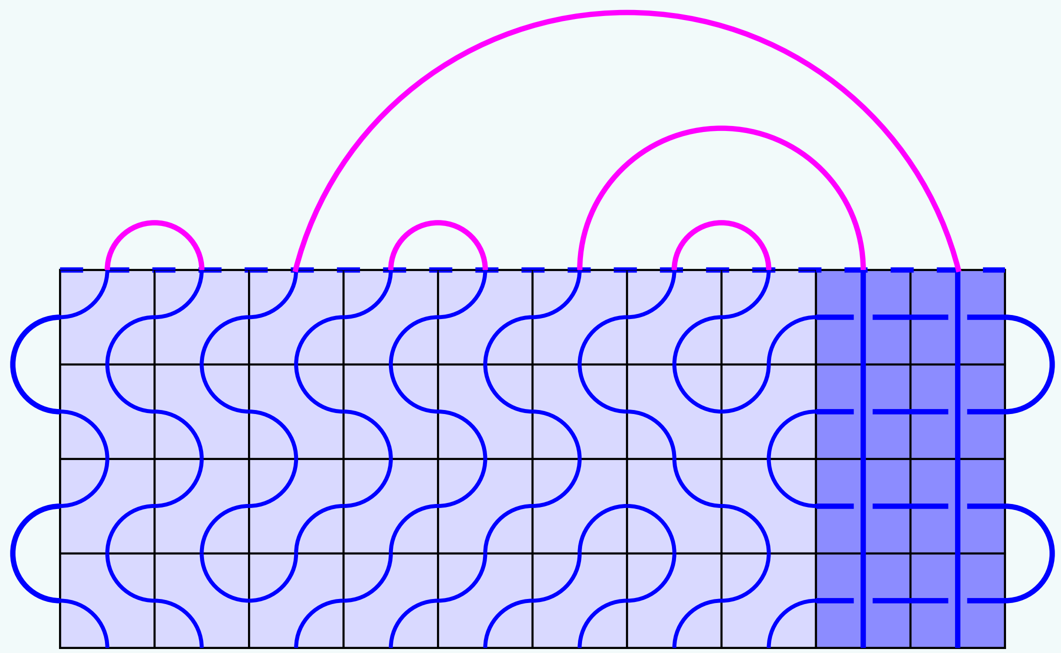
Defects

- More generally, the vector space of states $\mathcal{V}_N^{(\ell)}$ can contain ℓ *defects*:

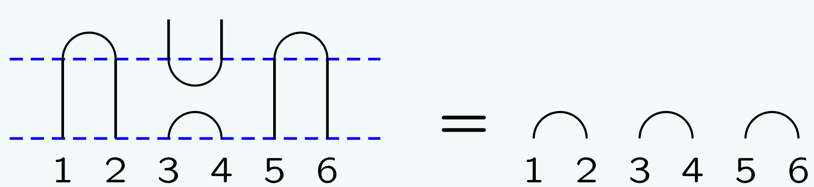
$N = 4,\ \ell = 2:$



- The ℓ defects can be closed on the right or the left. In this way, the number of defects propagating in the bulk is controlled by the boundary conditions. In particular, for $(1, s)$ boundary conditions, the $\ell = s - 1$ defects simply propagate along a boundary.



- Defects in the bulk can be annihilated in pairs but not created under the action of TL



etc.

- The transfer matrices are thus block-triangular with respect to the number of defects.

Dense Polymer Kac Table

- **Central charge:** $(p, p') = (1, 2)$

$$\boxed{c \;=\; 1 - \frac{6(p-p')^2}{pp'} \;=\; -2}$$

- **Infinitely extended Kac table of conformal weights:**

$$\boxed{\begin{aligned}\Delta_{r,s} \;&=\; \frac{(p'r-ps)^2-(p-p')^2}{4pp'}\\ &=\; \frac{(2r-s)^2-1}{8}, \qquad r,s=1,2,3,\ldots\end{aligned}}$$

- **Kac representation characters:**

$$\boxed{\chi_{r,s}(q) = q^{-c/24}\,\frac{q^{\Delta_{r,s}}(1-q^{rs})}{\prod_{n=1}^{\infty}(1-q^n)}}$$

- **Irreducible Representations:**

There is an irreducible representation for each distinct conformal weight. The Kac representations which happen to be irreducible are marked with a red quadrant.

s							
s	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
10	$\frac{63}{8}$	$\frac{35}{8}$	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\cdots$
9	6	3	1	0	0	1	$\cdots$
8	$\frac{35}{8}$	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\cdots$
7	3	1	0	0	1	3	$\cdots$
6	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\cdots$
5	1	0	0	1	3	6	$\cdots$
4	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	$\cdots$
3	0	0	1	3	6	10	$\cdots$
2	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	$\frac{99}{8}$	$\cdots$
1	0	1	3	6	10	15	$\cdots$
	1	2	3	4	5	6	r

Critical Percolation Kac Table

- **Central charge:** $(p, p') = (2, 3)$

$$\boxed{c \;=\; 1 - \frac{6(p-p')^2}{pp'} \;=\; 0}$$

- **Infinitely extended Kac table of conformal weights:**

$$\begin{aligned}\Delta_{r,s} &= \frac{(p'r - ps)^2 - (p-p')^2}{4pp'}\\ &= \frac{(3r-2s)^2 - 1}{24}, \qquad r, s = 1, 2, 3, \ldots\end{aligned}$$

- **Kac representation characters:**

$$\boxed{\chi_{r,s}(q) = q^{-c/24}\, \frac{q^{\Delta_{r,s}}(1-q^{rs})}{\prod_{n=1}^{\infty}(1-q^n)}}$$

- **Irreducible Representations:**

There is an irreducible representation for each distinct conformal weight. The Kac representations which happen to be irreducible are marked with a red quadrant.

s	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
10	12	$\frac{65}{8}$	5	$\frac{21}{8}$	1	$\frac{1}{8}$	$\cdots$
9	$\frac{28}{3}$	$\frac{143}{24}$	$\frac{10}{3}$	$\frac{35}{24}$	$\frac{1}{3}$	$-\frac{1}{24}$	$\cdots$
8	7	$\frac{33}{8}$	2	$\frac{5}{8}$	0	$\frac{1}{8}$	$\cdots$
7	5	$\frac{21}{8}$	1	$\frac{1}{8}$	0	$\frac{5}{8}$	$\cdots$
6	$\frac{10}{3}$	$\frac{35}{24}$	$\frac{1}{3}$	$-\frac{1}{24}$	$\frac{1}{3}$	$\frac{35}{24}$	$\cdots$
5	2	$\frac{5}{8}$	0	$\frac{1}{8}$	1	$\frac{21}{8}$	$\cdots$
4	1	$\frac{1}{8}$	0	$\frac{5}{8}$	2	$\frac{33}{8}$	$\cdots$
3	$\frac{1}{3}$	$-\frac{1}{24}$	$\frac{1}{3}$	$\frac{35}{24}$	$\frac{10}{3}$	$\frac{143}{24}$	$\cdots$
2	0	$\frac{1}{8}$	1	$\frac{21}{8}$	5	$\frac{65}{8}$	$\cdots$
1	0	$\frac{5}{8}$	2	$\frac{33}{8}$	7	$\frac{85}{8}$	$\cdots$
	1	2	3	4	5	6	r

Virasoro Representations and L_0

- In the continuum scaling limit, the transfer matrices give rise to a representation of the Virasoro algebra. Only L_0 is readily accessible from the lattice

$$\boldsymbol{D}(u) \sim e^{-u\mathcal{H}}, \qquad -\mathcal{H} \;\mapsto\; L_0 - \frac{c}{24}, \qquad Z_{r,s}(q) = \mathrm{Tr}\,\boldsymbol{D}(u)^P \mapsto q^{-c/24}\,\mathrm{Tr}\,q^{L_0} = \chi_{r,s}(q)$$

Type	Irreducible	Fully Reducible	Reducible yet Indecomposable	Decomposable
L_n	$\begin{pmatrix}{\color{red}\blacksquare}\end{pmatrix}$	$\begin{pmatrix}{\color{red}\blacksquare} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & {\color{red}\blacksquare} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & {\color{red}\blacksquare}\end{pmatrix}$	$\begin{pmatrix}{\color{red}\blacksquare} & {\color{gray}\blacksquare}\\ \mathbf{0} & {\color{red}\blacksquare}\end{pmatrix}$	$\begin{pmatrix}{\color{red}\blacksquare} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & {\color{red}\blacksquare} & {\color{gray}\blacksquare}\\ \mathbf{0} & \mathbf{0} & {\color{red}\blacksquare}\end{pmatrix}$
L_0	Diagonalizable	Diagonalizable	Jordan Cells of Rank ≥ 2	Jordan Cells

- **Rational Theories:**

  Irreducible representations are the building blocks for fusion. Fusion closes on the irreducible representations.

- **Logarithmic Theories:**

  Kac representations are the building blocks for fusion. Higher rank indecomposable representations arise from fusing Kac representations.

Lattice Fusion and Indecomposable Representations

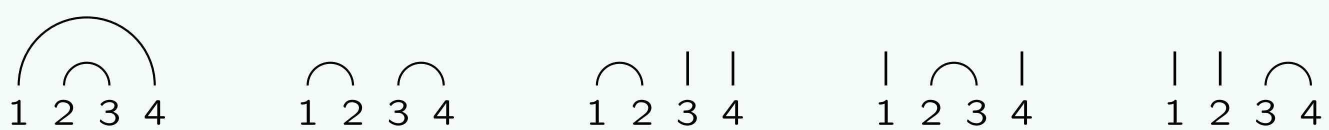
- For *Critical Dense Polymers*, the $(1,2)\otimes(1,2)=\left(-\frac{1}{8}\right)\otimes\left(-\frac{1}{8}\right)=0\;+\;0=(1,1)+(1,3)$ fusion yields an indecomposable representation. For $N=4$, the finitized partition function is ($q=$ modular parameter)

$$Z^{(N)}_{(1,2)|(1,2)}(q)=\underbrace{\chi^{(N)}_{(1,1)}(q)}_{0\text{ defects}}+\underbrace{\chi^{(N)}_{(1,3)}(q)}_{2\text{ defects}}=q^{-c/24}[(1+q^2)+(1+q+q^2)]=q^{-c/24}(2+q+2q^2)$$

- The Hamiltonian

$$\boldsymbol{D}(u)\sim e^{-u\mathcal{H}}\qquad -\mathcal{H}\;=\left(\begin{array}{cc|ccc} 0&1&0&0&0\\ 2&0&1&0&1\\ \hline 0&0&0&1&0\\ 0&0&1&0&1\\ 0&0&0&1&0\end{array}\right)+\sqrt{2}\,I\qquad -\mathcal{H}\;\mapsto L_0-\tfrac{c}{24}$$

acts on the five states with $\ell=0$ or $\ell=2$ defects



- The Jordan canonical form for $\mathcal{H}$ has rank 2 Jordan cells

$$-\mathcal{H}\sim\left(\begin{array}{cc|ccc} 0&0&1&0&0\\ 0&\sqrt{8}&0&0&1\\ \hline 0&0&0&0&0\\ 0&0&0&\sqrt{2}&0\\ 0&0&0&0&\sqrt{8}\end{array}\right)\sim\begin{pmatrix} 0&1&0&0&0\\ 0&0&0&0&0\\ 0&0&\sqrt{2}&0&0\\ 0&0&0&\sqrt{8}&1\\ 0&0&0&0&\sqrt{8}\end{pmatrix}\sim\begin{pmatrix} 0&1&0&0&0\\ 0&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&2&1\\ 0&0&0&0&2\end{pmatrix}=L_0^{(4)}$$

- The eigenvalues of $-\mathcal{H}$ approach the integer energies indicated in $L_0^{(4)}$ as $N\to\infty$.

Dense Polymer Virasoro Fusion Algebra

- The Virasoro fusion algebra of critical dense polymers $\mathcal{LM}(1,2)$ is

$$\boxed{\big\langle (2,1),(1,2)\big\rangle \;=\; \big\langle (r,1),(1,2k),\mathcal{R}_{2k};\;\; r,k\in\mathbb{N}\big\rangle}$$

- With the identifications $(k,2k')\equiv(k',2k)$, the fusion rules obtained empirically from the lattice are commutative, associative and agree with Gaberdiel and Kausch (1996)

$$\begin{aligned}
(r,1)\otimes(r',1) &= \bigoplus_{j=|r-r'|+1,\text{ by }2}^{r+r'-1} (j,1)\\
\hline
(1,2k)\otimes(1,2k') &= \bigoplus_{j=|k-k'|+1,\text{ by }2}^{k+k'-1} \mathcal{R}_{2j}\\
(1,2k)\otimes\mathcal{R}_{2k'} &= \bigoplus_{j=|k-k'|}^{k+k'} \delta^{(2)}_{j,\{k,k'\}}(1,2j)\\
\mathcal{R}_{2k}\otimes\mathcal{R}_{2k'} &= \bigoplus_{j=|k-k'|}^{k+k'} \delta^{(2)}_{j,\{k,k'\}}\mathcal{R}_{2j}\\
\hline
(r,1)\otimes(1,2k) &= \bigoplus_{j=|r-k|+1,\text{ by }2}^{r+k-1} (1,2j)=(r,2k)\\
(r,1)\otimes\mathcal{R}_{2k} &= \bigoplus_{j=|r-k|+1,\text{ by }2}^{r+k-1} \mathcal{R}_{2j}
\end{aligned}$$

s							
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
10	$\frac{63}{8}$	$\frac{35}{8}$	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\cdots$
9	6	3	1	0	0	1	$\cdots$
8	$\frac{35}{8}$	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\cdots$
7	3	1	0	0	1	3	$\cdots$
6	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\cdots$
5	1	0	0	1	3	6	$\cdots$
4	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	$\cdots$
3	0	0	1	3	6	10	$\cdots$
2	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	$\frac{99}{8}$	$\cdots$
1	0	1	3	6	10	15	$\cdots$
	1	2	3	4	5	6	r

$\mathcal{R}_{2k}=$ indecomposable $=(1,2k-1)\oplus_i(1,2k+1)$, $\qquad \delta^{(2)}_{j,\{k,k'\}}=2-\delta_{j,|k-k'|}-\delta_{j,k+k'}$

Extended Vacuum of Symplectic Fermions

- Critical dense polymers in the $\mathcal{W}$-extended picture is identified with *symplectic fermions*.
- The extended vacuum character of symplectic fermions is known to be

$$\hat{\chi}_{1,1}(q) = \sum_{n=1}^{\infty} (2n-1)\,\chi_{2n-1,1}(q)$$

This suggests the corresponding integrable boundary condition is the direct sum

$$\boxed{(1,1)_{\mathcal{W}} = \bigoplus_{n=1}^{\infty} (2n-1)\,(2n-1,1) \;=\; \mathcal{W}\text{-irreducible representation}}$$

- However, the BYBE is *not* linear and sums of solutions do *not* usually give new solutions. Rather, the *BYBE is closed under fusions*. If we can construct this direct sum from fusions, then automatically it will be a solution of the BYBE.

- Consider the triple fusion

$$\boxed{(2n-1,1)\otimes(2n-1,1)\otimes(2n-1,1) = (1,1)\oplus 3(3,1)\oplus 5(5,1)\oplus\cdots\oplus(2n-1)(2n-1,1)\oplus\cdots}$$

For large n, the coefficients stabilize and reproduce the extended vacuum $(1,1)_{\mathcal{W}}$. So the integrable boundary condition associated to the extended vacuum boundary condition is constructed by fusing three r-type integrable seams to the boundary

$$\boxed{(1,1)_{\mathcal{W}} := \lim_{n\to\infty}(2n-1,1)\otimes(2n-1,1)\otimes(2n-1,1) = \bigoplus_{n=1}^{\infty}(2n-1)\,(2n-1,1)}$$

Extended Boundary Conditions

- The extended vacuum $(1,1)_{\mathcal{W}}$ must act as the identity. In particular

$$(1,1)_{\mathcal{W}} \,\hat{\otimes}\, (1,1)_{\mathcal{W}} = (1,1)_{\mathcal{W}}$$

where $\hat{\otimes}$ denotes the fusion multiplication in the extended picture.

- The extended vacuum has the stability property

$$(2m-1,1) \otimes (1,1)_{\mathcal{W}} = (2m-1)\Big(\bigoplus_{n=1}^{\infty} (2n-1)\,(2n-1,1)\Big) = (2m-1)\,(1,1)_{\mathcal{W}}$$

- The extended fusion $\hat{\otimes}$ is therefore defined by

$$\boxed{(1,1)_{\mathcal{W}} \,\hat{\otimes}\, (1,1)_{\mathcal{W}} := \lim_{n\to\infty} \Big(\frac{1}{(2n-1)^3}(2n-1,1) \otimes (2n-1,1) \otimes (2n-1,1) \otimes (1,1)_{\mathcal{W}}\Big) = (1,1)_{\mathcal{W}}}$$

- The representation content is 4 $\mathcal{W}$-irreducible and 2 $\mathcal{W}$-indecomposable representations. Additional stability properties enable us to define

$$\boxed{\begin{aligned}
(1,s)_{\mathcal{W}} &:= (1,s) \otimes (1,1)_{\mathcal{W}} = \bigoplus_{n=1}^{\infty} (2n-1)\,(2n-1,s), \qquad s = 1,2\\
(2,s)_{\mathcal{W}} &:= \tfrac{1}{2}(2,s) \otimes (1,1)_{\mathcal{W}} = \bigoplus_{n=1}^{\infty} 2n\,(2n,s), \qquad s = 1,2\\
\hat{\mathcal{R}}_1 \equiv (\mathcal{R}_1)_{\mathcal{W}} &:= \mathcal{R}_1 \otimes (1,1)_{\mathcal{W}} = \bigoplus_{n=1}^{\infty} (2n-1)\,\mathcal{R}_{2n-1}\\
\hat{\mathcal{R}}_0 \equiv (\mathcal{R}_2)_{\mathcal{W}} &:= \tfrac{1}{2}\mathcal{R}_2 \otimes (1,1)_{\mathcal{W}} = \bigoplus_{n=1}^{\infty} 2n\,\mathcal{R}_{2n}
\end{aligned}}$$

Cayley Table of Symplectic Fermion Fusion Rules

- The $\mathcal{W}$-extended fusion rules follow from the Virasoro fusion rules combined with stability. The extended fusion rules and characters agree with Gaberdiel and Runkel (2007):

$\hat{\otimes}$	0	1	$-\frac{1}{8}$	$\frac{3}{8}$	$\hat{\mathcal{R}}_0$	$\hat{\mathcal{R}}_1$
0	0	1	$-\frac{1}{8}$	$\frac{3}{8}$	$\hat{\mathcal{R}}_0$	$\hat{\mathcal{R}}_1$
1	1	0	$\frac{3}{8}$	$-\frac{1}{8}$	$\hat{\mathcal{R}}_1$	$\hat{\mathcal{R}}_0$
$-\frac{1}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\hat{\mathcal{R}}_0$	$\hat{\mathcal{R}}_1$	$2(-\frac{1}{8})+2(\frac{3}{8})$	$2(-\frac{1}{8})+2(\frac{3}{8})$
$\frac{3}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\hat{\mathcal{R}}_1$	$\hat{\mathcal{R}}_0$	$2(-\frac{1}{8})+2(\frac{3}{8})$	$2(-\frac{1}{8})+2(\frac{3}{8})$
$\hat{\mathcal{R}}_0$	$\hat{\mathcal{R}}_0$	$\hat{\mathcal{R}}_1$	$2(-\frac{1}{8})+2(\frac{3}{8})$	$2(-\frac{1}{8})+2(\frac{3}{8})$	$2\hat{\mathcal{R}}_0+2\hat{\mathcal{R}}_1$	$2\hat{\mathcal{R}}_0+2\hat{\mathcal{R}}_1$
$\hat{\mathcal{R}}_1$	$\hat{\mathcal{R}}_1$	$\hat{\mathcal{R}}_0$	$2(-\frac{1}{8})+2(\frac{3}{8})$	$2(-\frac{1}{8})+2(\frac{3}{8})$	$2\hat{\mathcal{R}}_0+2\hat{\mathcal{R}}_1$	$2\hat{\mathcal{R}}_0+2\hat{\mathcal{R}}_1$

Example: Consider the extended fusion rule $1\,\hat{\otimes}\,1 = 0$:

$$\begin{aligned}
(2,1)_{\mathcal{W}}\,\hat{\otimes}\,(2,1)_{\mathcal{W}} &:= \Big(\tfrac{1}{2}(2,1)\otimes(1,1)_{\mathcal{W}}\Big)\hat{\otimes}\Big(\tfrac{1}{2}(2,1)\otimes(1,1)_{\mathcal{W}}\Big)\\
&= \tfrac{1}{4}\Big((2,1)\otimes(2,1)\Big)\otimes\Big((1,1)_{\mathcal{W}}\,\hat{\otimes}\,(1,1)_{\mathcal{W}}\Big)\\
&= \tfrac{1}{4}\Big((1,1)\oplus(3,1)\Big)\otimes(1,1)_{\mathcal{W}} = \tfrac{1}{4}(1+3)(1,1)_{\mathcal{W}} = (1,1)_{\mathcal{W}}
\end{aligned}$$

$\mathcal{W}$-Representation Content of $\mathcal{LM}(p,p')$

	$\mathcal{LM}(p,p')$	Symplectic Fermions	Critical Percolation
$\mathcal{W}$-reps	$6pp' - 2p - 2p'$	6	26
Rank 1	$2p + 2p' - 2$	4	8
Rank 2	$4pp' - 2p - 2p'$	2	14
Rank 3	$2(p-1)(p'-1)$	0	4
$\mathcal{W}$-irred chars	$2pp' + \frac{1}{2}(p-1)(p'-1)$	4	13

- Kac tables of 4 and 13 $\mathcal{W}$-irreducible representations for symplectic fermions and critical percolation:

s		
2	$-\frac{1}{8}$	$\frac{3}{8}$
1	0	1
	1	2 $\quad r$

s		
3	$\frac{1}{3}, \frac{10}{3}$	$-\frac{1}{24}, \frac{35}{24}$
2	$1, 5$	$\frac{1}{8}, \frac{21}{8}$
1	$(0)\,2, 7$	$\frac{5}{8}, \frac{33}{8}$
	1	2 $\quad r$

$\mathcal{W}$-Characters of Critical Percolation

$$\hat{\chi}_0(q) = 1$$

$$\hat{\chi}_1(q) = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} k^2 \Big[q^{(12k-7)^2/24} - q^{(12k+1)^2/24}\Big]$$

$$\hat{\chi}_2(q) = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} k^2 \Big[q^{(12k-5)^2/24} - q^{(12k-1)^2/24}\Big]$$

$$\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

$$\hat{\chi}_5(q) = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} k(k+1) \Big[q^{(12k-1)^2/24} - q^{(12k+7)^2/24}\Big]$$

$$\hat{\chi}_7(q) = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} k(k+1) \Big[q^{(12k+1)^2/24} - q^{(12k+5)^2/24}\Big]$$

$$\hat{\chi}_{\frac{1}{3}}(q) = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} (2k-1) q^{3(4k-3)^2/8}$$

$$\hat{\chi}_{\frac{10}{3}}(q) = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} 2k\, q^{3(4k-1)^2/8}$$

$$\hat{\chi}_{\frac{1}{8}}(q) = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} (2k-1)\, q^{(6k-5)^2/6}$$

$$\hat{\chi}_{\frac{5}{8}}(q) = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} (2k-1)\, q^{(6k-4)^2/6}$$

$$\hat{\chi}_{\frac{21}{8}}(q) = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} 2k\, q^{(6k-2)^2/6}$$

$$\hat{\chi}_{\frac{33}{8}}(q) = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} 2k\, q^{(6k-1)^2/6}$$

$$\hat{\chi}_{-\frac{1}{24}}(q) = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} (2k-1)\, q^{(6k-6)^2/6}$$

$$\hat{\chi}_{\frac{35}{24}}(q) = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} 2k\, q^{(6k-3)^2/6}$$

- These agree with Feigin, Gainutdinov, Semikhatov and Tipunin (2005).

Summary

- Critical dense polymers and critical percolation are the first two members $\mathcal{LM}(1,2)$ and $\mathcal{LM}(2,3)$ of the Yang-Baxter integrable series of logarithmic minimal models $\mathcal{LM}(p,p')$.

- In the Virasoro picture, there is an infinity of integrable boundary conditions labelled by $r,s = 1,2,3,\ldots$. In the continuum scaling limit, these give rise to so-called Kac representations (r,s) of the Virasoro algebra. These representations are organized into an infinitely extended Kac table and are not in general irreducible.

- Fusion of two representations is implemented on the lattice by taking the integrable boundary conditions associated to the representations on the left and right boundaries of the strip. Fusion of the Kac representations (r,s) can give rise to reducible but indecomposable representations with Jordan cells of rank 2 or 3.

- The Virasoro fusion rules for $\mathcal{LM}(p,p')$ have been obtained empirically by studying fusion on the lattice. These fusion rules are closed, commutative and associative.

- Critical dense polymers in the extended $\mathcal{W}$-algebra picture is identified with *symplectic fermions*.

- For $\mathcal{LM}(p,p')$ in the extended $\mathcal{W}$-algebra picture, the infinity of Virasoro representations are reorganized into a finite number of $\mathcal{W}$-representations that close amongst themselves under fusion. The $\mathcal{W}$ fusion rules are obtained from the Virasoro fusion rules.

- It remains an open problem to find the Verlinde formula for $\mathcal{LM}(p,p')$ with $p > 1$.

Chiral Symplectic Fermions (Kausch 1995)

- The central charge of symplectic fermions is $c = -2$ and the stress-energy tensor is

$$\boxed{T(z) \;=\; \sum_{n\in\mathbb{Z}} L_n\, z^{-n-2} \;=\; \tfrac{1}{2}\, d_{\alpha\beta} :\!\chi^\alpha(z)\chi^\beta(z)\!:}$$

where $d_{\alpha\beta}$ is the inverse of the anti-symmetric tensor $d^{\alpha\beta}$ with $\alpha, \beta = \pm$.

- The chiral algebra $\mathcal{W}$ is generated by a two-component fermion field

$$\chi^\alpha(z) \;=\; \sum_{n\in\mathbb{Z}} \chi^\alpha_n\, z^{-n-1}, \qquad \alpha = \pm$$

of conformal weight $\Delta = 1$. The modes satisfy the anticommutation relations

$$\{\chi^\alpha_m, \chi^\beta_n\} \;=\; m\, d^{\alpha\beta}\, \delta_{m,-n}$$

- Alternatively, the extended symmetry algebra $\mathcal{W}$ is generated by the Virasoro modes L_n and the modes of a triplet of weight 3 fields W^a_n.

Logarithmic Ising and Yang-Lee Kac Tables

s							
s	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
10	$\frac{225}{16}$	$\frac{161}{16}$	$\frac{323}{48}$	$\frac{65}{16}$	$\frac{33}{16}$	$\frac{35}{48}$	$\cdots$
9	11	$\frac{15}{2}$	$\frac{14}{3}$	$\frac{5}{2}$	1	$\frac{1}{6}$	$\cdots$
8	$\frac{133}{16}$	$\frac{85}{16}$	$\frac{143}{48}$	$\frac{21}{16}$	$\frac{5}{16}$	$-\frac{1}{48}$	$\cdots$
7	6	$\frac{7}{2}$	$\frac{5}{3}$	$\frac{1}{2}$	0	$\frac{1}{6}$	$\cdots$
6	$\frac{65}{16}$	$\frac{33}{16}$	$\frac{35}{48}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{35}{48}$	$\cdots$
5	$\frac{5}{2}$	1	$\frac{1}{6}$	0	$\frac{1}{2}$	$\frac{5}{3}$	$\cdots$
4	$\frac{21}{16}$	$\frac{5}{16}$	$-\frac{1}{48}$	$\frac{5}{16}$	$\frac{21}{16}$	$\frac{143}{48}$	$\cdots$
3	$\frac{1}{2}$	0	$\frac{1}{6}$	1	$\frac{5}{2}$	$\frac{14}{3}$	$\cdots$
2	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{35}{48}$	$\frac{33}{16}$	$\frac{65}{16}$	$\frac{323}{48}$	$\cdots$
1	0	$\frac{1}{2}$	$\frac{5}{3}$	$\frac{7}{2}$	6	$\frac{55}{6}$	$\cdots$
	1	2	3	4	5	6	r

s	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
10	$\frac{27}{5}$	$\frac{91}{40}$	$\frac{2}{5}$	$-\frac{9}{40}$	$\frac{2}{5}$	$\frac{91}{40}$	$\cdots$
9	4	$\frac{11}{8}$	0	$-\frac{1}{8}$	1	$\frac{27}{8}$	$\cdots$
8	$\frac{14}{5}$	$\frac{27}{40}$	$-\frac{1}{5}$	$\frac{7}{40}$	$\frac{9}{5}$	$\frac{187}{40}$	$\cdots$
7	$\frac{9}{5}$	$\frac{7}{40}$	$-\frac{1}{5}$	$\frac{27}{40}$	$\frac{14}{5}$	$\frac{247}{40}$	$\cdots$
6	1	$-\frac{1}{8}$	0	$\frac{11}{8}$	4	$\frac{63}{8}$	$\cdots$
5	$\frac{2}{5}$	$-\frac{9}{40}$	$\frac{2}{5}$	$\frac{91}{40}$	$\frac{27}{5}$	$\frac{391}{40}$	$\cdots$
4	0	$-\frac{1}{8}$	1	$\frac{27}{8}$	7	$\frac{95}{8}$	$\cdots$
3	$-\frac{1}{5}$	$\frac{7}{40}$	$\frac{9}{5}$	$\frac{187}{40}$	$\frac{44}{5}$	$\frac{567}{40}$	$\cdots$
2	$-\frac{1}{5}$	$\frac{27}{40}$	$\frac{14}{5}$	$\frac{247}{40}$	$\frac{54}{5}$	$\frac{667}{40}$	$\cdots$
1	0	$\frac{11}{8}$	4	$\frac{63}{8}$	13	$\frac{155}{8}$	$\cdots$
	1	2	3	4	5	6	r