Topological Discrete Algebra in Topological Orders

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Topological order

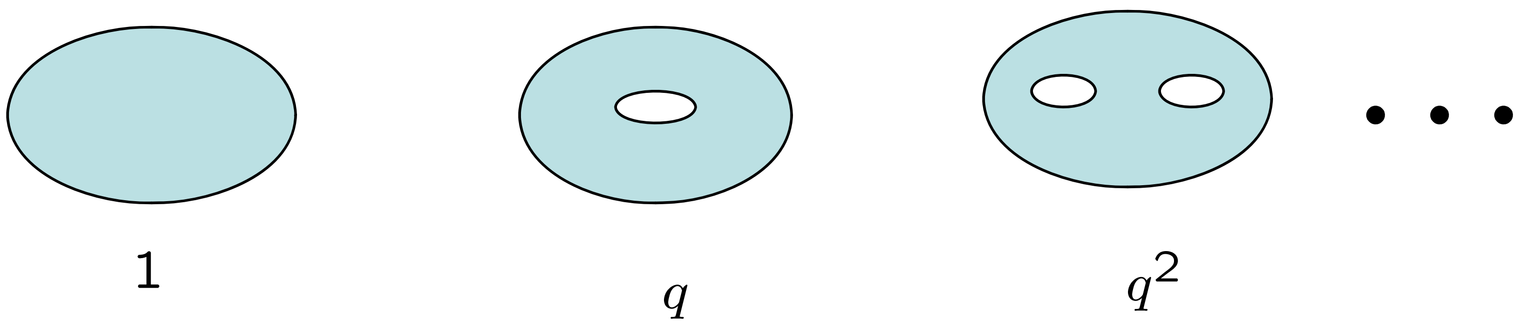
◆ Topological orders are conventionally characterized by the ground-state degeneracy depending on topology of the space (= topological degeneracy)

Wen '90

A typical example is fractional quantum Hall systems

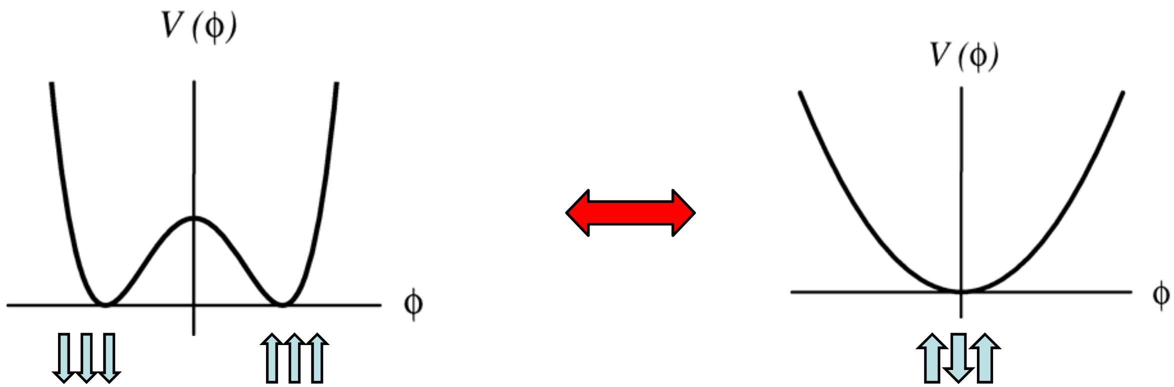
ex.) Laughlin state $\nu = \frac{1}{q}$

GS degeneracy



Tao-Wu '84, Wen-Niu '90

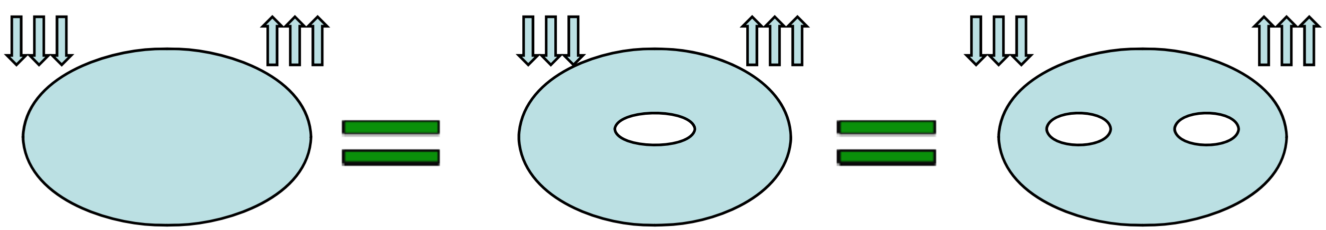
The ground state degeneracy is useful even for symmetry breaking orders



ordered
2-fold degeneracy

not ordered
no degeneracy

But, the degeneracy is **independent of the topology !**



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At present, several different systems are known to exhibit topological orders.

- boson system
- fermion system
- in the presence or in the absence of magnetic filed (without or with time-reversal invariance)
- 2+1 and 3+1 dim. system

X.G.Wen ‘91,
Read-Sachdev ‘91,
Senthil-Fisher ‘00,
Moessner-Sondhi ‘01,
Misguichi-Serban-Pesquir ‘02,
Balents-Fisher-Girvin ‘02,
Motrunich-Senthil ‘02,
Kitaev,
Lawler-Kee-Y.-B .Kim-Vishwanath ’08, ….

Common characteristics

① All the known models have an excitation with fractional charge.

Oshikawa-Senthil '06

But, some models have the following other interesting characteristics

② Some models have an excitation with fractional statistics or non-abelian statistics.

③ Some models show fractional quantum Hall effects.

All these fractionalization can be treated in a unified way in terms of braid group and large gauge transformation.

Fractionalization = Topological Order

Topological Discrete Algebra

Outline

① Introduction
② Topological discrete algebra
(Hidden symmetry, Heisenberg algebra, 't Hooft algebra)
③ Ground state degeneracy
④ FQHE

MS, M.Kohmoto, Y.S.Wu, PRL '06

⑤ Generalization to non-abelian gauge theories in 3+1 dim
& quark (de)confinement

MS, PRD '08

② Topological Discrete Algebra (d=2)

MS, M.Kohmoto, Y.S.Wu, PRL '06

Our assumptions are the following

definition of charge

① The system is on a torus.
② There exists U(1) symmetry.
③ The system is gapped.
④ Charge fractionalization occurs.

In other words, we assume that
there exists a quasi-particle with fractional charge

$$e^* = \frac{p}{q} e$$

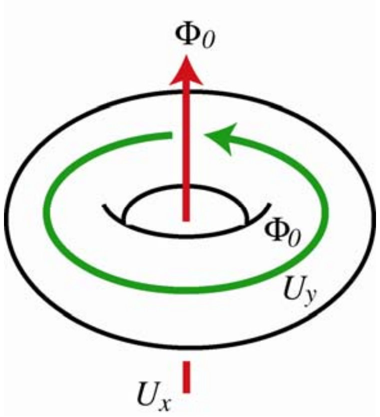
p, q coprime integers

(e charge of the constituent particle)

We start with the following non-trivial processes on a torus

(Wu-Hatsugai-Kohmoto '91, Oshikawa-Senthil '06)

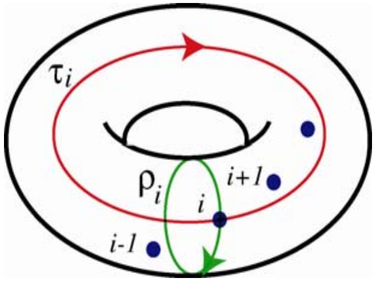
a. adiabatic unit flux insertions through holes of torus



$$\boxed{U_x,\ U_y}$$

$$\Phi_0 = \frac{2\pi\hbar}{e}$$ a unit flux

b. translations of i-th quasi-particle along loops of torus



$$\boxed{\rho_i,\ \tau_i}$$

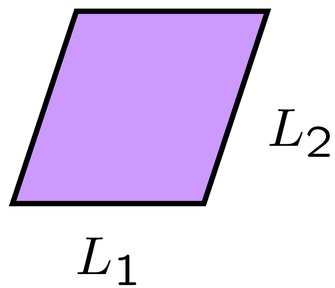
◆Note that the unit flux insertions are unitary equivalent to the following adiabatic changes of the boundary conditions.

Consider the twisted boundary condition on the torus

phase

$$\boxed{\begin{aligned} \psi_n(x_i + L_1) &= e^{i\theta_x}\psi_n(x_i) \\ \psi_n(y_i + L_2) &= e^{i\theta_y}\psi_n(y_i) \end{aligned}}$$

phase



By the large gauge transformation

$$U(\Phi_x, \Phi_y) = e^{i(e\Phi_x/L_1)(x_1+\cdots+x_N)+i(e\Phi_y/L_2)(y_1+\cdots+y_N)}$$

1. the flux is reduced as $\quad -i\dfrac{\partial}{\partial x_i} \to -i\dfrac{\partial}{\partial x_i} - e\dfrac{\Phi_x}{L_1}$

2. the boundary conditions are changed as

$$U(\Phi_x, \Phi_y, x_i+L_1)\psi(x_i+L_1) = e^{i(e\Phi_x+\theta_x)}U(\Phi_x, \Phi_y, x_i)\psi(x_i)$$

Using the unitary transformation, we can delete the inserted unit flux completely, but the boundary condition parameters θ_x and θ_y change by one period 2π.

◆ Thus U_x (U_y) is unitary equivalent to adiabatic change of θ_x (θ_y) by 2π

$$\boxed{\begin{aligned}\psi_n(x_i+L_1)&=e^{i\theta_x}\psi_n(x_i)\\ \psi_n(y_i+L_2)&=e^{i\theta_y}\psi_n(y_i)\end{aligned}}$$

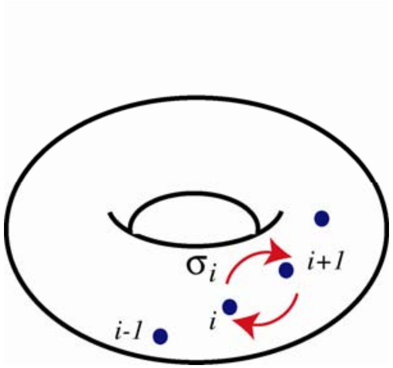
$$\boxed{\begin{aligned}\theta_x &\xrightarrow{U_x} \theta_x+2\pi\\ \theta_y &\xrightarrow{U_y} \theta_y+2\pi\end{aligned}}$$

The spectrum should be invariant under U_x and U_y

⇒ **U_x and U_y ･･･ a kind of symmetry**

We also consider the exchange of quasi-particles.

c. exchange between i-th and (i+1)-th quasi-particles



$\boxed{\sigma_i}$

The reason why we take into account this operation is that the translations along the loops are not independent of the exchanges of quasi-particles. Indeed they form the braid group algebra.

Braid Group on torus

Birman '69, Einarsson '90

$$A_{j,i} \equiv \tau_j^{-1} \rho_i \tau_j \rho_i^{-1}, \quad C_{j,i} \equiv \rho_j^{-1} \tau_i \rho_j \tau_i^{-1}, \quad (1 \le i < j \le N),$$

$$\sigma_k \sigma_l = \sigma_l \sigma_k, \quad (1 \le k \le N-3,\ |l-k| \ge 2),$$
$$\sigma_k \sigma_{k+1} \sigma_k = \sigma_{k+1} \sigma_k \sigma_{k+1}, \quad (1 \le k \le N-2),$$
$$\underline{\tau_{i+1} = \sigma_i^{-1} \tau_i \sigma_i^{-1}}, \quad \underline{\rho_{i+1} = \sigma_i \rho_i \sigma_i},$$
$$\tau_1 \sigma_j = \sigma_j \tau_1, \quad \rho_1 \sigma_j = \sigma_j \rho_1, \quad \underline{\sigma_i^2 = A_{i+1,i}},$$
$$(1 \le i \le N-1,\ 2 \le j \le N-1)$$

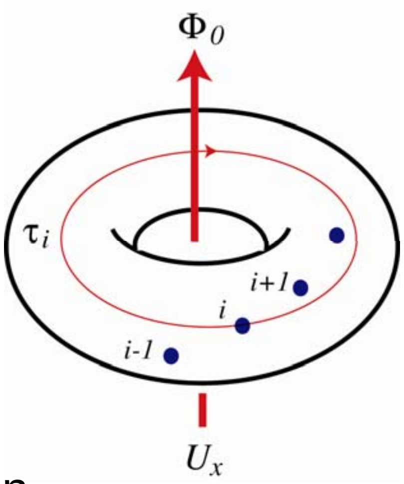
$$A_{m,l} \tau_k = \tau_k A_{m,l}, \quad A_{m,l} \rho_k = \rho_k A_{m,l},$$
$$\tau_i \tau_j = \tau_j \tau_i, \quad \rho_i \rho_j = \rho_j \rho_i,$$
$$C_{j,i} = (\tau_i \tau_j) A_{j,i}^{-1} (\tau_j^{-1} \tau_i^{-1}), \quad A_{j,i} = (\rho_i \rho_j) C_{j,i}^{-1} (\rho_j^{-1} \rho_i^{-1}),$$
$$C_{j,i} = (A_{j,j-1}^{-1} \cdots A_{j,i+1}^{-1}) A_{j,i}^{-1} (A_{j,i+1} \cdots A_{j,j-1}),$$
$$\tau_1 \rho_1 \tau_1^{-1} \rho_1^{-1} = A_{2,1} A_{3,1} \cdots A_{N-1,1} A_{N,1}$$
$$(1 \le k < l < m \le N,\ 1 \le i < j \le N)$$

The commutation relations between the braid group operators and the flux insertions are determined by AB effect.

From AB effect, we have

$$\boxed{U_x \tau_i = \underline{e^{-2\pi i p/q}} \tau_i U_x}$$

$$e^* = \frac{p}{q} e$$



AB phase

translation after flux insertion

In a similar manner, we obtain

$$U_y \rho_i = e^{-2\pi i p/q} \rho_i U_y$$

$$U_x \rho_i = \rho_i U_x, \quad U_x \sigma_i = \sigma_i U_x,$$
$$U_y \tau_i = \tau_i U_y, \quad U_y \sigma_i = \sigma_i U_y.$$

On the other hand, the commutation relation between the flux insertion operators is determined by Schur's lemma

$U_x U_y U_x^{-1} U_y^{-1}$ commutes with all the braid group operators

Schur's lemma

$$\Rightarrow \quad U_x U_y U_x^{-1} U_y^{-1} = const = e^{2\pi i \lambda}$$

$$\boxed{U_x U_y = e^{2\pi i \lambda} U_y U_x}$$

Furthermore, U_x^q (and U_y^q) commutes with all the braid group operators

$$\Rightarrow \quad \lambda = \frac{k}{l}$$

k, l: coprime integers

l: a diviser of q $\quad (e^* = \frac{p}{q} e)$

Thus, our tool to examine the topological order is the following.

① Braid Group

② AB effect

$$U_x\tau_i = e^{-2\pi ip/q}\tau_i U_x, \qquad U_y\rho_i = e^{-2\pi ip/q}\rho_i U_y$$
$$U_x\rho_i = \rho_i U_x, \qquad U_x\sigma_i = \sigma_i U_x,$$
$$U_y\tau_i = \tau_i U_y, \qquad U_y\sigma_i = \sigma_i U_y.$$

③ Schur's lemma

$$U_x U_y = e^{2\pi i\lambda} U_y U_x, \qquad \lambda = k/l$$

Topological discrete algebra (1)

If the quasi-particle obeys abelian statistics, the algebra is simplified.

$$\sigma_i = e^{i\theta}\mathbf{1}, \text{ (boson, fermion, anyon)}$$

The solution of braid group is

$$\tau_j = e^{-2i\theta(j-1)}T_x, \quad \rho_j = e^{2i\theta(j-1)}T_y$$

$$T_xT_y = e^{-2i\theta}T_yT_x, \quad \theta = \frac{m}{n}\pi$$

m, n coprime integers

$$T_xT_y = e^{-2\pi i m/n}T_yT_x, \quad U_xU_y = e^{2\pi i\lambda}U_yU_x,$$

$$U_xT_xU_x^{-1} = e^{-2\pi i p/q}T_x, \quad U_yT_xU_y^{-1} = T_x,$$

$$U_xT_yU_x^{-1} = T_y, \quad U_yT_yU_y^{-1} = e^{-2\pi i p/q}T_y$$

Topological discrete algebra (2)

$$\theta = \pi\frac{m}{n}, \; e^* = \frac{p}{q}e, \; \lambda = \frac{k}{l}$$

The statistical property is naturally combined with the charge fractionalization in terms of the topological discrete algebra.

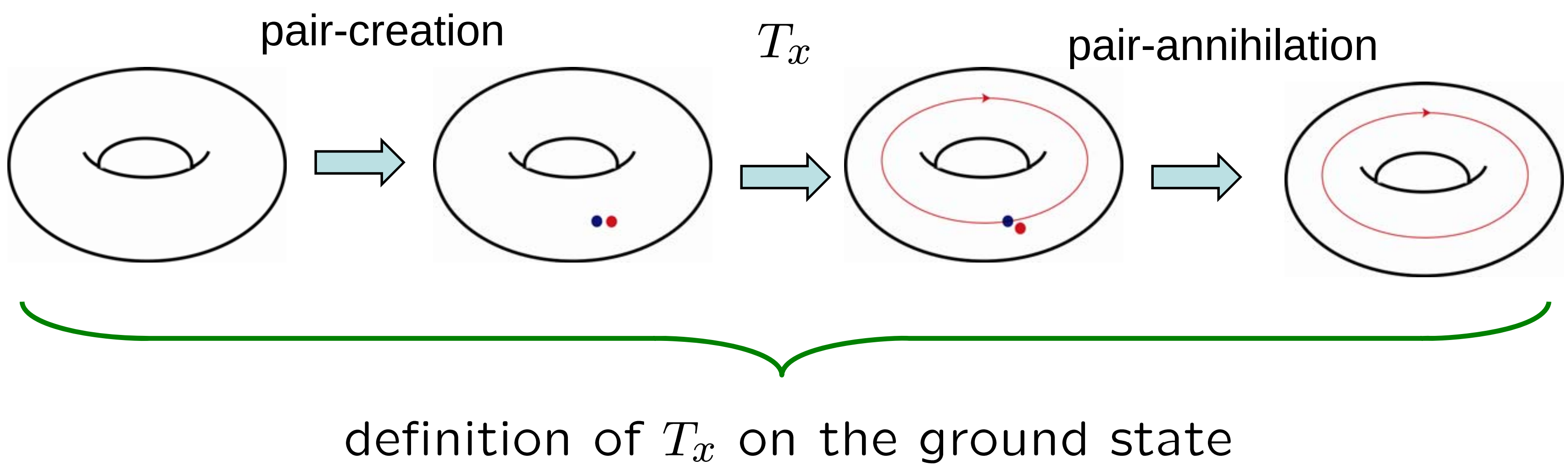
Questions:

1. Can the topological discrete algebra explain the ground state degeneracy ?

2. What is the physical meaning of the fractional parameter λ ?

④ Ground state degeneracy

To count ground state degeneracy, we define T_x on the ground state



In a similar manner, we define T_y on the ground state

● We have unitary operators which act on the ground state and they have non-commutative relations. This implies that the ground state must be degenerate.

Let us take the basis of the ground state to be an eigenstate of T_x

$$T_x|\eta\rangle = e^{i\eta}|\eta\rangle$$

then, we have n different eigenstates

$$\Rightarrow \quad T_x(T_y^s|\eta\rangle) = e^{i(\eta - 2\pi sm/n)}T_y^s|\eta\rangle \qquad s = 1, \cdots, n$$

n-fold degeneracy

$$(\theta = \pi \frac{m}{n})$$

We can also take the basis to be an eigenstate of T_x and T_y^n

$$T_x T_y^n = T_y^n T_x$$

$$T_x|\eta_1, \eta_2\rangle = e^{i\eta_1}|\eta_1, \eta_2\rangle, \quad T_y^n|\eta_1, \eta_2\rangle = e^{i\eta_2}|\eta_1, \eta_2\rangle$$

In this case, we have qQ different eigenstates

$$\Rightarrow \quad \begin{aligned} T_x(U_x^s U_y^t|\eta_1, \eta_2\rangle) &= e^{i(\eta_1 + 2\pi sp/q)} U_x^s U_y^t|\eta_1, \eta_2\rangle \\ T_y^n(U_x^s U_y^t|\eta_1, \eta_2\rangle) &= e^{i(\eta_2 + 2\pi tnp/q)} U_x^s U_y^t|\eta_1, \eta_2\rangle \end{aligned}$$

qQ eigenvalues $\quad (s = 1, \cdots, q,\ t = 1, \cdots, \mathcal{Q},\ n/q = \mathcal{N}/\mathcal{Q}),$

$\mathcal{N}$ and $\mathcal{Q}$: co-prime integers

qQ-fold degeneracy $\quad (\theta = \pi \frac{m}{n},\ e^* = \frac{p}{q} e),$

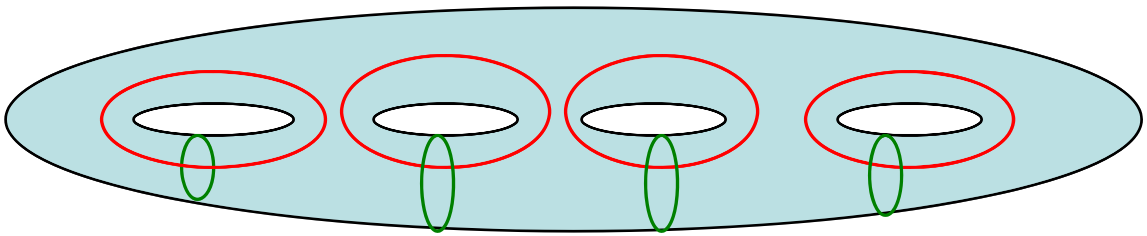
The minimal ground state degeneracy is the least common multiple of n and $qQ=nQ^2/N$

($n/q = \mathcal{N}/\mathcal{Q}, \mathcal{N}, \mathcal{Q}$: co-prime integers)

$\boxed{n\mathcal{Q}^2\text{-fold ground state degeneracy}}$

In general

$\theta = \pi\frac{m}{n},\ e^* = \frac{p}{q}e$



$(n\mathcal{Q}^2)^g$-fold degeneracy

- Ground state degeneracy is obtained from charge fractionalization and fractional statistics
- It depends on the topology of the space

This formula reproduces known results for GS degeneracy.

ex.)

1) Laughlin state $\nu = \frac{1}{q}$ $e^* = \frac{e}{q}, \quad \theta = \frac{\pi}{q}$

$n = q$, thus
$\mathcal{N} = \mathcal{Q} = 1$ $\Rightarrow$ $\boxed{\text{The minimal degeneracy} = q^g}$

$(n\mathcal{Q}^2)^g$-fold degeneracy

It reproduces the Wen-Niu's result

On a torus, the minimal degeneracy is realized by

$$T_x = S_{q\times q}, \quad T_y = R_{q\times q}, \quad U_x = R^{-1}_{q\times q}, \quad U_y = S_{q\times q}$$

$$S_{q\times q} = \mathrm{diag}\{1, e^{2\pi i/q}, \dots, e^{2\pi i(q-1)/q}\}$$

$$R_{q\times q} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

$$S_{q\times q}R_{q\times q} = e^{-2\pi i/q} R_{q\times q}S_{q\times q}$$

$$\Longrightarrow \quad U_x U_y = e^{-2\pi i/q} U_y U_x \qquad \lambda = -\frac{1}{q}$$

2) When the quasi-particle is charge fractionalized boson or fermion,..

$$e^* = \frac{p}{q}e, \qquad \theta = 0, \pi$$

$\mathcal{N} = 1, \mathcal{Q} = q$ $\Rightarrow$ minimal degeneracy= q^{2g}

$(n\mathcal{Q}^2)^g$-fold degeneracy

Oshikawa-Senthil '06
(Kitaev model,
quantum dimer model)

The minimal degeneracy is realized when

$$T_x = R_{q\times q} \otimes 1_{q\times q}, \qquad T_y = 1_{q\times q} \otimes R_{q\times q},$$
$$U_x = S^p_{q\times q} \otimes 1_{q\times q}, \qquad U_y = 1_{q\times q} \otimes S^p_{q\times q}$$

$\Rightarrow$ $U_x U_y = U_y U_x$ $\qquad \lambda = 1$

Topological degeneracy is explained by the topological discrete algebra.

⑤ The physical meaning of λ

To consider the physical meaning of λ, we calculate the Hall conductance by using the linear response theory

$$\sigma_{xy} = \frac{e^2}{h}\overbrace{\frac{1}{d}\sum_{K=1}^{d}}^{\text{average over degeneracy}} \underbrace{\frac{1}{2\pi i}\int_0^{2\pi} d\theta_x \int_0^{2\pi}}_{\text{average over the boundary conditions}} d\theta_y \left[\langle \frac{\partial \phi_K}{\partial \theta_y} | \frac{\partial \phi_K}{\partial \theta_x} \rangle - \langle \frac{\partial \phi_K}{\partial \theta_x} | \frac{\partial \phi_K}{\partial \theta_y} \rangle \right]$$

Niu-Thouless-Wu '84

Here $\phi_K(\theta_x, \theta_y)$ is the ground state with the twisted boundary condition (K degeneracy: $K = 1, \cdots, d$.)

$$\phi_K(x_i + L_1) = e^{i\theta_x}\phi_K(x_i)$$
$$\phi_K(y_i + L_2) = e^{i\theta_y}\phi_K(y_i)$$

The ground state satisfies

$$U_a|\phi_K(\vec{\theta})\rangle = e^{i\gamma_a(\vec{\theta})}|\phi_K(\vec{\theta}+2\pi\hat{e}_a)\rangle$$

$$\gamma_a(\vec{\theta}) = i\int_{\theta_a}^{\theta_a+2\pi}\langle\phi_K|\frac{\partial}{\partial\theta_a}|\phi_K\rangle$$

From $U_xU_y = e^{2\pi i\lambda}U_yU_x$

$$\boxed{\gamma_x(\vec{\theta}+2\pi\hat{e}_y)+\gamma_y(\vec{\theta}) = \gamma_y(\vec{\theta}+2\pi\hat{e}_x)+\gamma_x(\vec{\theta})+2\pi\lambda+2\pi M}$$

M: integer

Using this, we have

$$\begin{aligned}
\sigma_{xy} &= \frac{e^2}{h}\int_0^{2\pi}\int_0^{2\pi}\frac{d\theta_x d\theta_y}{2\pi i}\left[\frac{\partial}{\partial\theta_y}\langle\phi_K|\frac{\partial}{\partial\theta_x}|\phi_K\rangle-(\theta_x\leftrightarrow\theta_y)\right]\\
&= -\frac{e^2}{h}\left[\int_0^{2\pi}\frac{d\theta_y}{2\pi}\frac{\partial\gamma_x(0,\theta_y)}{\partial\theta_y}-(\theta_x\leftrightarrow\theta_y)\right]\\
&= -\frac{e^2}{h}[\lambda+M]
\end{aligned}$$

⇒ Fractional λ implies the fractional quantum Hall effect

Part 1.

Summary

◆ Using the braid group formulation, we found a closed algebraic structure which characterizes topological orders.

◆ Topological degeneracy is due to the topological discrete algebra.

cf.) For symmetry breaking orders

σ_x

The degeneracy is due to broken generators.

◆The fractional quantum Hall effect is a result of the non-commutative structure of the flux insertion operations

Part 2.

⑤ Generalization to Non-Abelian Gauge Theories & Quark confinement

M.S. Phys. Rev D (08)

Idea

The Quark has **fractional charges**

The quark deconfinement implies the topological order ?

yes

But.. the quark is an **elementary particle**, not a collective excitation.

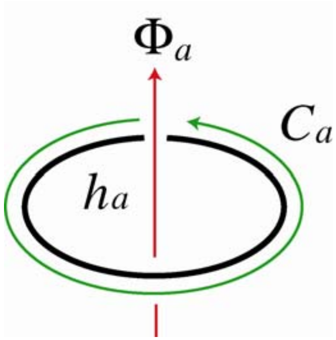
Nevertheless, non-trivial topological discrete algebra can be constructed in a similar manner ..

An important difference

$$T^3 = S^1 \otimes S^1 \otimes S^1$$

Due to the existence of gluons, the unit flux is reduced to $\Phi_a = 2\pi/3e$

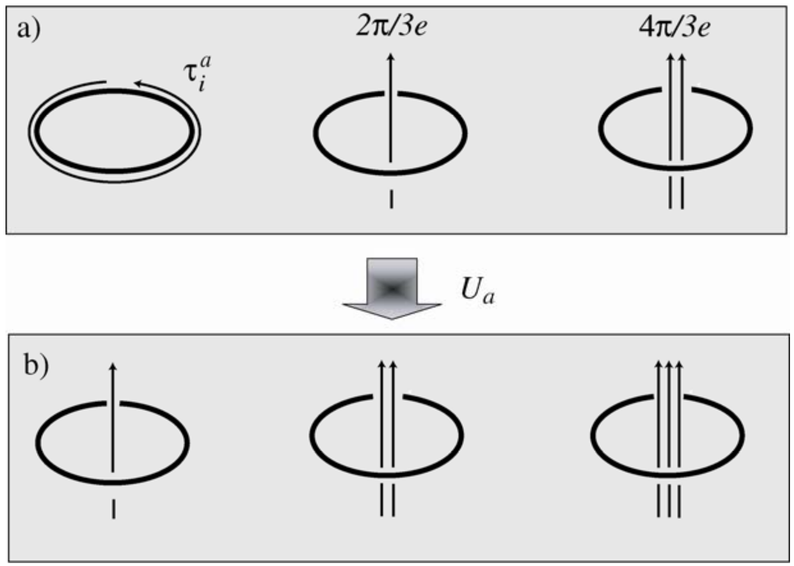
e: the minimal charge of the constitute particle (charge of down quark)



cf.) For electron system, $\Phi_a = 2\pi/e$

Due to gluon fluctuations, there exist center vortices of SU(3) in each holes of three dimensional torus.

$$T^3 = S^1 \otimes S^1 \otimes S^1$$

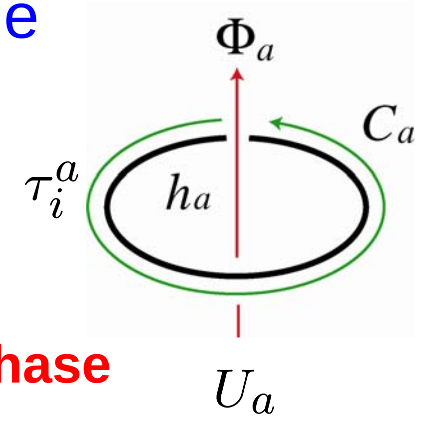


Thus, the physics is the same after the flux insertion by $2\pi/3e$

(not $2\pi/e$)

The unit flux is reduced to $\Phi_a = 2\pi/3e$

We have different Aharanov-Bohm phases between quark deconfinement phase and confinement one



- If the excitation is **quark**, we have

$$\tau_i^a U_a = \underline{e^{-2\pi i/3}} U_a \tau_i^a$$

translation after flux insertion

AB phase

- If the excitation is **hadron,** we have

$$\tau_i^a U_a = U_a \tau_i^a$$

No AB phase

Other commutation relations are determined by ..

② Permutation group

$$\sigma_k^2 = 1, \quad 1 \le k \le N-1,$$
$$(\sigma_k \sigma_{k+1})^3 = 1, \quad 1 \le k \le N-1$$
$$\sigma_k \sigma_l = \sigma_l \sigma_k, \quad 1 \le k \le N-3, \quad |l-k| \ge 2,$$
$$\tau_{i+1}^a = \sigma_i \tau_i^a \sigma_i, \quad 1 \le i \le N-1, \quad a = 1,2,3,$$
$$\tau_1^a \sigma_j = \sigma_j \tau_1^a, \quad 2 \le j \le N, \quad a = 1,2,3,$$
$$\tau_i^a \tau_j^b = \tau_j^b \tau_i^a, \quad i,j = 1, \cdots N, \quad a,b = 1,2,3.$$

③ Shur's lemma

- For quark, ⇨ $U_a U_b U_a^{-1} U_b^{-1} = e^{2\pi \lambda_{a,b}} \quad U_a^3 = \mathrm{const.}$

- For hadron, ⇨ $U_a = \mathrm{const.}$

In 3+1 dim, the excitations (quarks or hadrons) are boson or fermion,

$$\sigma_i = \pm\mathbf{1}.$$

The unique solution of the permutation group

$$\Rightarrow \quad \boxed{\tau_i^a = T_a \text{ with } T_a T_b = T_b T_a}$$

We have two different topological discrete algebras

For quark deconfinement phase

T_a: quark winding operator

$$T_a U_b = e^{-(2\pi i/3)\delta_{a,b}} U_b T_a \qquad U_a U_b = e^{2\pi i \lambda_{a,b}} U_b U_a$$

$$T_a T_b = T_b T_a \qquad U_a^3 = \text{const.}$$

⇕

For quark confinement phase

T_a: hadron winding operator

$$T_a T_b = T_b T_a \qquad U_a = \text{const.}$$

trivial !

If quarks are **deconfined**, the physical states are classified with the permutation group of **quarks**.

Non trivial topological discrete algebra

$$T_a U_b = e^{-(2\pi i/3)\delta_{a,b}} U_b T_a \qquad U_a U_b = e^{2\pi i \lambda_{a,b}} U_b U_a$$

$$T_a T_b = T_b T_a \qquad U_a^3 = \text{const.}$$

Topological ground state degeneracy !

On the other hand, …

If quarks are **confined**, the physical states are classified with the permutation group of **hadrons**.

⇓

Topological discrete algebra is trivial

⇓

No topological degeneracy !

The confinement and deconfinement phases in QCD are discriminated by the topological ground state degeneracy !

For SU(N) QCD on $T^n \times R^{4-n}$

- deconfinement: N^n –fold ground state degeneracy
- confinement: No topological degeneracy

Test of our argument for quark confinement

- Wilson's criterion
- 1-loop analysis
- Witten index
- Fradkin-Shenker's phase diagram

⇨ All of them are consistent with our argument

Summary

- A hidden symmetry in topological discrete algebra can be explicitly constructed in terms of the braid group (or permutation group) and flux insertions.

- **Fractionalization = Topological Order**

- **Quark deconfinement = Topological Order**