Quantum Entanglement in Exactly Solvable Models

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Outline

1. Introduction

a) Reduced density matrix & Entanglement entropy (EE)

b) Previous works in quantum spin chains

Theoretical laboratory

Quantum entanglement

in solvable models

Non-local/topological properties of quantum many-body ground state

- Edge state/fractionalization
- Fractional exclusion statistics

2. Entanglement in the Affleck-Keneddy-Lieb-Tasaki Model

a) Generic valence-bond-solid (VBS) state in 1D
b) Exact Expression for EE and Edge-state Picture

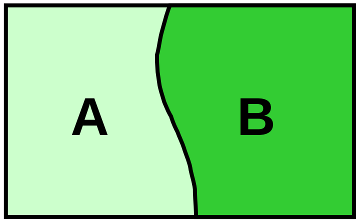
3. Particle-Entanglement in the Calogero-Sutherland (CS) Model

a) CS model and Fractional Exclusion Statistics
b) L-particle Reduced Density Matrix & EE based on Particle Partitioning

Entanglement Entropy (1)

Definition of Entanglement Entropy(EE)

- Many body ground state (assumed to be unique) : $|G\rangle$
- Density matrix : $\rho_{AB} = |G\rangle\langle G|$
- Reduced density matrix : $\rho_A = \mathrm{Tr}_B \rho_{AB}$
- Entanglement Entropy : $S_A = -\mathrm{Tr}_A\, \rho_A \log_2 \rho_A$



subsystems

Physical Meaning of the EE

- Direct product state

$$|\Psi\rangle_{AB} = |\phi\rangle_A |\psi\rangle_B \Longrightarrow |\phi\rangle_A\langle\phi| \Longrightarrow \mathcal{S}_A = 0$$

Quantify entanglement between A and B.

- Maximally entangled state

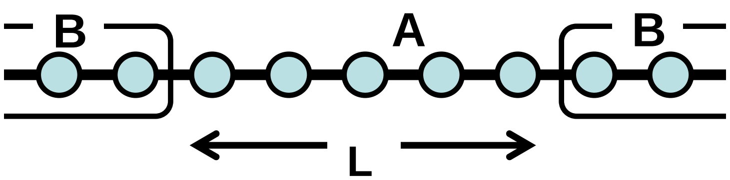
$$|\Psi\rangle_{AB} = \frac{1}{\sqrt{D}} \sum_{j=1}^{D} |\phi_j\rangle_A |\psi_j\rangle_B \Longrightarrow \frac{1}{D} \sum_{j=1}^{D} |\phi_j\rangle_A \langle\phi_j| \Longrightarrow \mathcal{S}_A = \log_2 D$$

Orthonormal.

Entanglement Entropy in Spin Chains

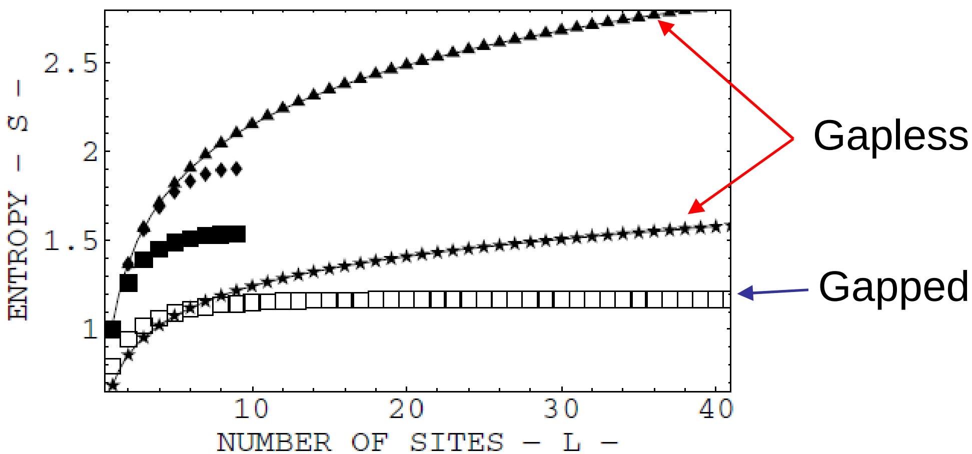
EE in the XXZ and XY models with magnetic field

- Vidal et al., PRL 90 (2003)



$$H_{XXZ} = \sum_{l=0}^{N-1} (\sigma_l^x \sigma_{l+1}^x + \sigma_l^y \sigma_{l+1}^y + \Delta \sigma_l^z \sigma_{l+1}^z - \lambda \sigma_l^z),$$

$$H_{XY} = -\sum_{l=0}^{N-1} \left(\frac{a}{2} [(1+\gamma) \sigma_l^x \sigma_{l+1}^x + (1-\gamma) \sigma_l^y \sigma_{l+1}^y] + \sigma_l^z \right).$$



□ : XY(a=1.1, γ=1)

■ : XXZ(Δ=2.5, λ=0)

★ : XY(a=1, γ=1)

▲ : XY(a=∞, γ=0)

◆ : XXZ(Δ=1, λ=0)

critical

Gapless point : $S_L \sim \frac{c+\bar{c}}{6} \log_2 L + k$ (Conformal field theory prediction)

Conjecture: gapped→saturation, gapless→logarithmic divergence

EE in the S=1 valence-bond-solid(VBS) state

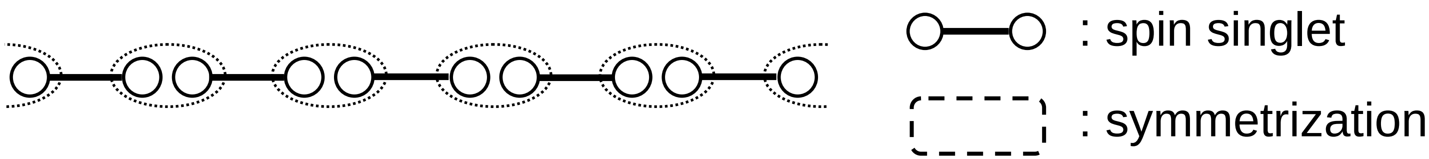
Haldane gap ('83): S=integer AFM Heisenberg chains are gapped.

Exp. → $Ni(C_2H_8N_2)_2NO_2(ClO_4)$ (NENP) etc.,

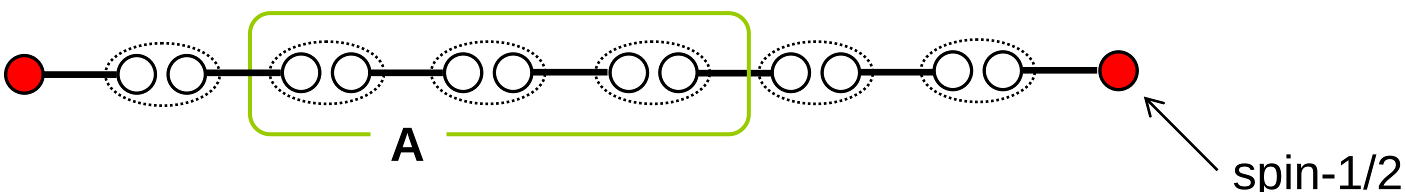
Affleck-Kennedy-Lieb-Tasaki(AKLT) model('87)

$$H = J\sum_j \vec{S}_j\cdot\vec{S}_{j+1}+\frac{1}{3}(\vec{S}_j\cdot\vec{S}_{j+1})^2 = J\sum_j 2P^{S=2}_{j,j+1}-\frac{2}{3}$$

GS → S=1 VBS state



• Fan, Korepin and Roychowdhury, PRL 94 ('04)



1. Exact expression for EE.
2. $S_L \to \log_2 4 = 2\log_2 2$ in the large block size limit.

→ 'Partial' confirmation of the conjecture.

Outline

1. Introduction

2. Entanglement in the AKLT Model

a) Generic valence-bond-solid (VBS) state in 1D

b) Exact expression for EE & Edge-state picture

c) VBS on an arbitrary graph

Related papers: 1. H. K., T. Hirano & Y. Hatsugai, Phys. Rev. B **76** (2007).

2. H. K., T. Hirano & V. E. Korepin, J. Phys. A **41** (2008).

3. Y. Xu, H. K., T. Hirano & V. E. Korepin, J. Stat. Phys **133** (2008).

4. Y. Xu, H. K., T. Hirano & V. E. Korepin, Q. Info. Proc. **7**, (2008).

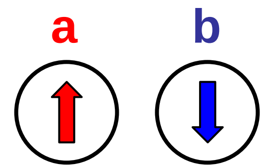
3.Entanglement in the Calogero-Sutherland (CS) Model

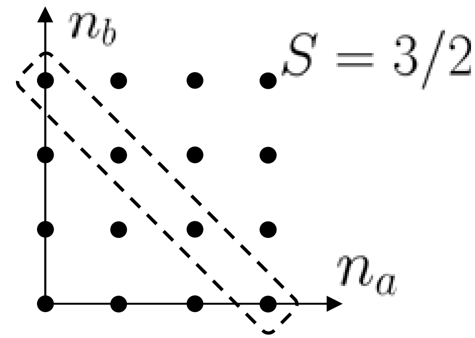
VBS with arbitrary integer-spin S

Scwinger boson(SB) rep. of spin operator

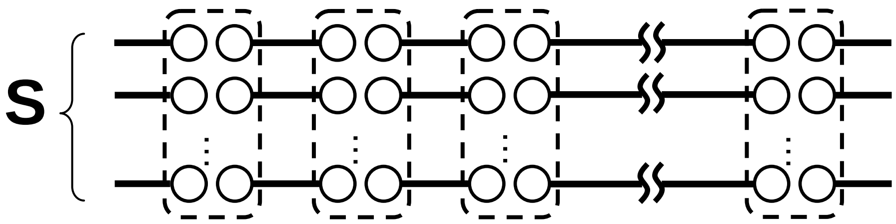
$$\begin{cases} S_j^+ = a_j^\dagger b_j,\ S_j^- = b_j^\dagger a_j, \\ S_j^z \ = \ (a_j^\dagger a_j \ - \ b_j^\dagger b_j)/2, \end{cases}$$

Constraint : $a_j^\dagger a_j + b_j^\dagger b_j = 2S.$



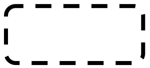


Construction of S=integer VBS state



: spin singlet

$(a_j^\dagger b_{j+1}^\dagger - b_j^\dagger a_{j+1}^\dagger)$



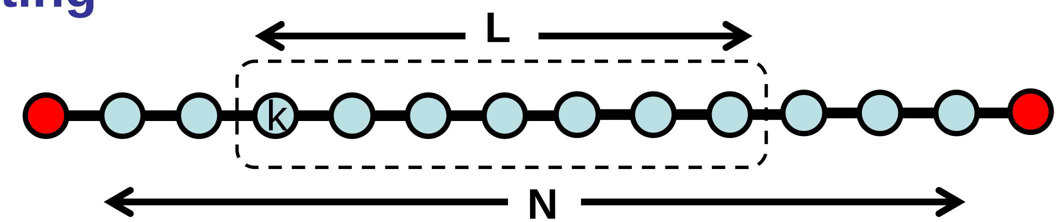
: symmetrization

SB rep.: $$|\mathrm{VBS}\rangle = \prod_{j=0}^{L} (a_j^\dagger b_{j+1}^\dagger - b_j^\dagger a_{j+1}^\dagger)^S |\mathrm{vac}\rangle$$

Corresponding Hamiltonian (Arovas-Auerbach-Haldane)

Zero energy GS of $$H = \sum_{j=1}^{N-1} \sum_{J=S+1}^{2S} A_J P_{j,j+1}^J + \pi_{0,1} + \pi_{N,N+1},$$

Setting



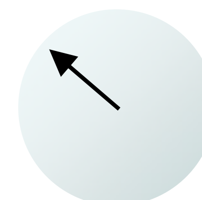
: Spin S site

: Spin S/2 site

- Density matrix : $\rho = |\mathrm{VBS}\rangle\langle\mathrm{VBS}|/\langle\mathrm{VBS}|\mathrm{VBS}\rangle,$
- Reduced density matrix (RDM) : $\rho_L = \mathrm{Tr}_{\overline{\mathcal{B}_L}}\rho$
- Entanglement Entropy (EE) : $\mathcal{S}_L = -\mathrm{Tr}_{\mathcal{B}_L}\rho_L \log_2 \rho_L$

Coherent State

$$\hat{\Omega} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$



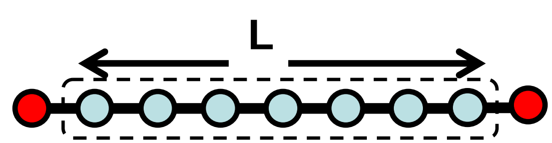
$$|\hat{\Omega}\rangle = \frac{(ua^\dagger + vb^\dagger)^{2S}}{\sqrt{(2S)!}}|0\rangle, \qquad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos(\theta/2)e^{i\phi/2} \\ \sin(\theta/2)e^{-i\phi/2} \end{pmatrix}$$

$$\mathrm{Tr}\mathcal{O} = \frac{2S+1}{4\pi}\int d\hat{\Omega}\langle\hat{\Omega}|\mathcal{O}|\hat{\Omega}\rangle$$

The constraint $a_j^\dagger a_j + b_j^\dagger b_j = 2S$ is automatically satisfied!

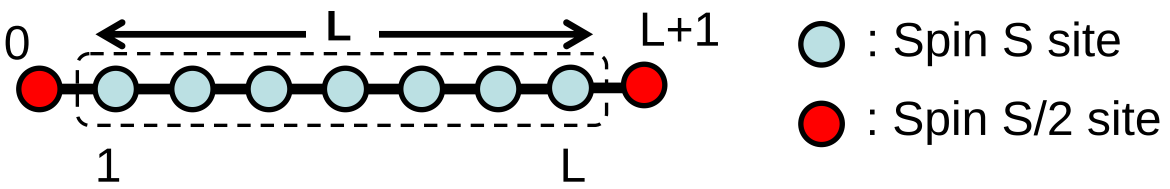
Theorem :

The RDM ρ_L does not depend on both the starting site k and the total length N.



We can set N=L without loss of generality.

• N=L

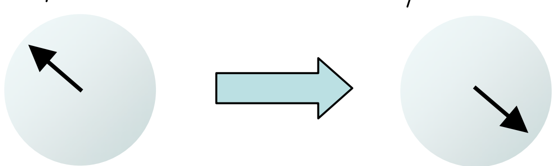


Important Property 1: $\mathcal{S}_A = \mathcal{S}_B$ (∵ Schmidt decomposition)

- Reduced density matrix of two end spin-S/2's : $\rho_{\hat{L}}$
- Entanglement Entropy : $\mathcal{S}_L = \mathcal{S}_{\hat{L}}$

Important Property 2: transformation property of spinors

$|\hat{\Omega}\rangle$ (θ, ϕ) $|-\hat{\Omega}\rangle$ $(\pi-\theta, \phi+\pi)$



(u, v) $(iv^*, -iu^*)$

$$P_0^\dagger|0\rangle = (a_0^\dagger v_1^* - b_0^\dagger u_1^*)^S|0\rangle$$
$$= (-i)^S\sqrt{S!}\,|-\hat{\Omega}_1\rangle$$

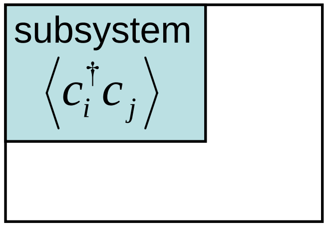
Coherent state rep. of $\rho_{\hat{L}}$ ***= Density matrix(DM)-DM correlation***

$$\rho_{\hat{L}} = \frac{\int \prod_{j=1}^{L} \frac{d\hat{\Omega}_j}{4\pi} \prod_{k=1}^{L-1} \left(\frac{1-\hat{\Omega}_k\cdot\hat{\Omega}_{k+1}}{2}\right)^S |\hat{\Omega}_1\rangle_0\langle\hat{\Omega}_1| \otimes |\hat{\Omega}_L\rangle_{L+1}\langle\hat{\Omega}_L|}{\int \left(\prod_{j=1}^{L} \frac{d\hat{\Omega}_j}{4\pi}\right) \prod_{k=1}^{L-1} \left(\frac{1-\hat{\Omega}_k\cdot\hat{\Omega}_{k+1}}{2}\right)^S}.$$

Transfer matrix

EE and correlation functions :Physical Meaning

cf) Free fermionic



Matrix elements of $\rho_{\hat{L}}$ are completely determined by the two point correlation functions such as $\langle \vec{S}_1 \cdot \vec{S}_L \rangle$.

Eigenvalues of $\rho_{\hat{L}}$

- Transfer matrix: $T_{k,k+1} = \left(\frac{1 - \hat{\Omega}_k \cdot \hat{\Omega}_{k+1}}{2} \right)^S$

$$T_{k,k+1} = \frac{4\pi}{S+1} \sum_{l=0}^{S} \lambda(l) \sum_{m=-l}^{l} Y_l^m(\hat{\Omega}_k) \overline{Y_l^m(\hat{\Omega}_{k+1})}$$

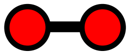
$$\lambda(l) = \frac{(-1)^l S!(S+1)!}{(S-l)!(S+l+1)!}$$

$$\rho_{\hat{L}} = \frac{4\pi}{(S+1)^2} \sum_{l=0}^{S} \lambda(l)^{L-1} \sum_{m=-l}^{l} [T_l^{(m)} \otimes (T_l^{(m)})^\dagger]$$

Spherical tensor

$$\frac{2 \cdot S/2+1}{4\pi} \int d\hat{\Omega} |\hat{\Omega}\rangle Y_l^m(\hat{\Omega}) \langle \hat{\Omega}|$$

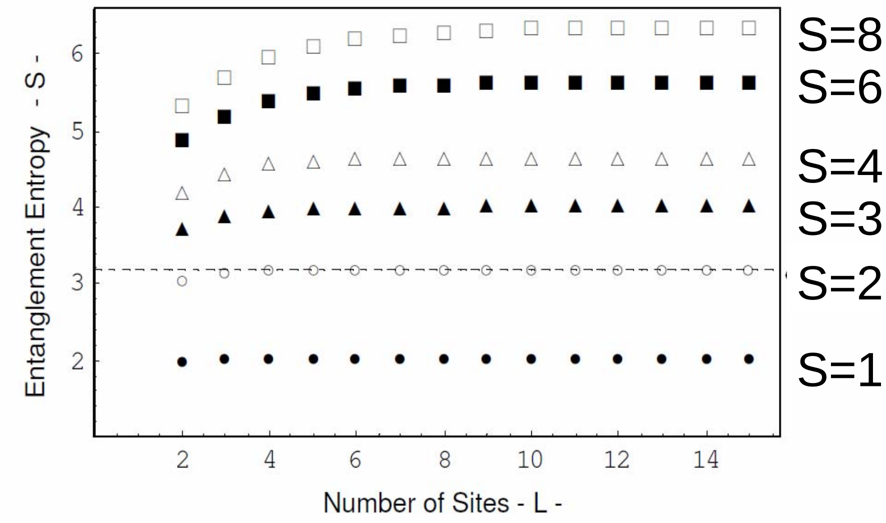
$$\rho_{\hat{L}}(J) = \frac{4\pi}{(S+1)^2} \sum_{l=0}^{S} \lambda(l)^{L-1} \, I_l \left(\frac{1}{2} J(J+1) - \frac{S}{2} \left(\frac{S}{2} + 1 \right) \right)$$

Total spin at boundaries  $J(= 0, 1, 2, \ldots, S)$

$I_j(X)$ $\quad I_0(X) = \frac{1}{4\pi}, \; I_1(X) = \frac{3}{4\pi} \frac{X}{(S/2+1)^2}$,…

Recursion relation found by Freitag & Muller-Hartman('91)

Entanglement Entropy



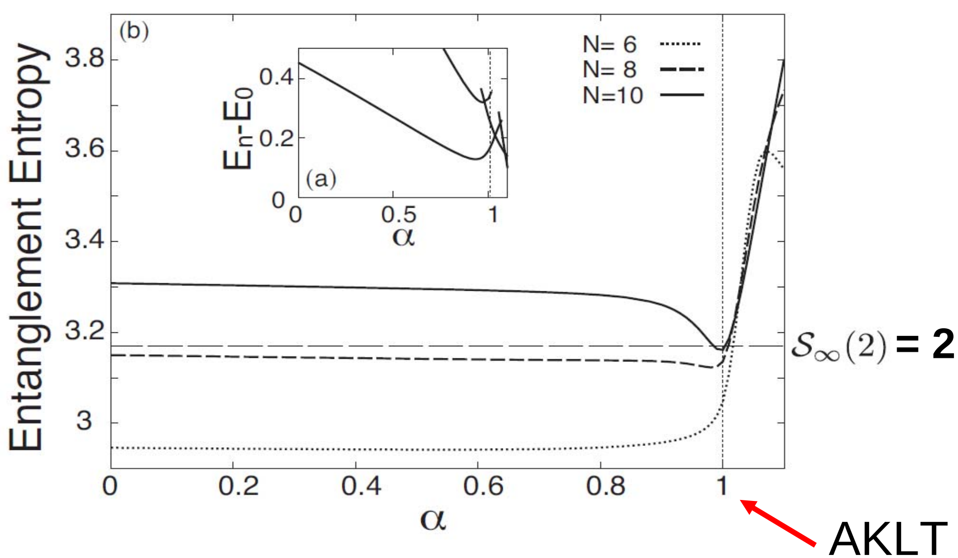
$$\mathcal{S}_L = -\sum_{J=0}^{S}(2J+1)\rho_{\hat{L}}(J)\log_2\rho_{\hat{L}}(J)$$

Saturation value: $2\log_2(S+1)$

Degrees of freedom

$$(2\cdot S/2+1)^2 = (S+1)^2$$

EE from Exact Diagonalizations



$\mathcal{S}_\infty(2)$ = **2log$_2$ 3**

- **S=2 Spin Hamiltonian**

$$H = \sum_{i=1}^{N}\vec{S}_i\cdot\vec{S}_{i+1}$$

$$+\alpha\left\{\frac{2}{9}(\vec{S}_i\cdot\vec{S}_{i+1})^2+\frac{1}{63}(\vec{S}_i\cdot\vec{S}_{i+1})^3+\frac{10}{7}\right\}$$

$\alpha = 0$: Heisenberg point,

$\alpha = 1$: AKLT point

- **S=1 Spin Hamiltonian**

(JPSJ, ('07) (Hirano & Hatsugai))

EE & Edge states in VBS (non-universal term has a physical meaning)

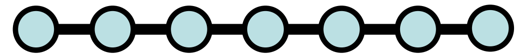
Saturation value: $2\log_2(S+1)$

Degrees of freedom

$$(2\cdot S/2+1)^2=(S+1)^2$$

Coincides with # of edge states!

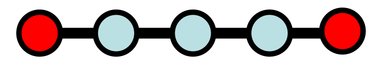
Degeneracy (open boundary)



Not an accident!

- ***The RDM ρ_L is exactly spanned by the edge states.***

$$\rho_L=\left[\frac{S+1}{(2S+1)!}\right]^L\frac{(S+1)}{(4\pi)^2}\int \mathrm{d}\hat{\Omega}_0\mathrm{d}\hat{\Omega}_{L+1}B^\dagger|\mathrm{VBS}_L\rangle\langle\mathrm{VBS}_L|B$$



$$=\left[\frac{S+1}{(2S+1)!}\right]^L\frac{S!S!}{S+1}\sum_{J=0}^{S}\sum_{M=-J}^{J}|\mathrm{VBS}_L(J,M)\rangle\langle\mathrm{VBS}_L(J,M)|.$$

$$B^\dagger=\left(u_0^*b_1^\dagger-v_0^*a_1^\dagger\right)^S\left(a_L^\dagger v_{L+1}^*-b_L^\dagger u_{L+1}^*\right)^S,$$

$$|\mathrm{VBS}_L(J,M)\rangle=\prod_{j=1}^{L-1}(a_j^\dagger b_{j+1}^\dagger-b_j^\dagger a_{j+1}^\dagger)^S|J,M\rangle_{1,L}|\mathrm{vac}\rangle_{2,\dots,L-1}$$

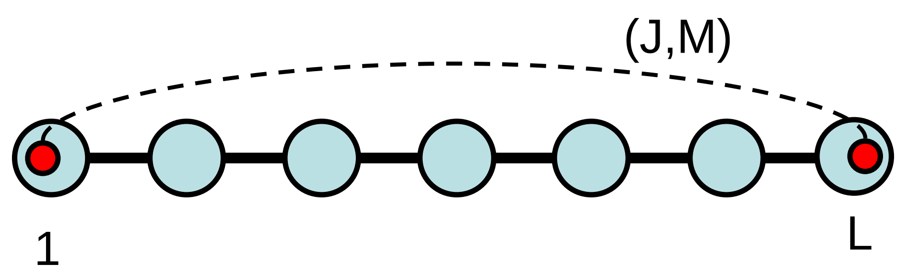
Projection onto degenerate g.s. of the block Hamiltonian.

$$H_b=\sum_{j=1}^{L-1}\sum_{J=S+1}^{2S}C_JP_{j,j+1}^J$$

Edge-state basis

$|\mathrm{VBS}_L(J,M)\rangle$

:mutually orthogonal



- $|\mathrm{VBS}_L(J,M)\rangle$ are mutually orthogonal. ← Symmetry!
- $|\mathrm{VBS}_L(J,M)\rangle$ are eigenstates of the total spin operators for "**SUBSYSTEM**":

$$[S^{\pm}_{\mathrm{tot}}, \prod_{j=1}^{L-1}(a_j^\dagger b_{j+1}^\dagger - b_j^\dagger a_{j+1}^\dagger)^S] = 0, \quad [S^{z}_{\mathrm{tot}}, \prod_{j=1}^{L-1}(a_j^\dagger b_{j+1}^\dagger - b_j^\dagger a_{j+1}^\dagger)^S] = 0.$$

- Calculation of EE → Calculation of the norm of edge states.

$$\mathcal{S}_L = \sum_{J=0}^{S}\sum_{M=-J}^{J}\Lambda(J,M)\log_2\Lambda(J,M) \qquad \Lambda(J,M) = \frac{\langle \mathrm{VBS}_L(J,M)|\mathrm{VBS}_L(J,M)\rangle}{\langle \mathrm{VBS}|\mathrm{VBS}\rangle}$$

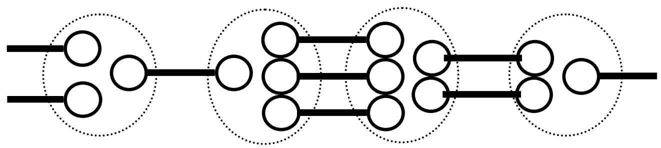
The norm is given by the Wigner 3-j symbols! (Indepedent of M)

$$\begin{aligned}&\langle \mathrm{VBS}_L(J,M)|\mathrm{VBS}_L(J,M)\rangle \\ = \quad & \frac{(2J+1)!((2S+1)!)^L}{(S+1)^{L-1}(S+J+1)!(S-J+1)!(J+1)!(J+1)!}\sum_{l_1=0}^{S}\sum_{l_L=0}^{S-J}\sum_{l=0}^{J} \\ & (2l_1+1)(2l_L+1)(2l+1)\lambda^{L-1}(l_1,S)\lambda(l_L,S-J)\lambda^2(l,J)\begin{pmatrix} l_1 & l_L & l \\ 0 & 0 & 0\end{pmatrix}^2.\end{aligned}$$

- ***By-product***: We obtain the explicit form of the e.v. of RDM and EE ***without*** using the recursion relation found by Freitag & Muller-Hartmann.

Other Related Works

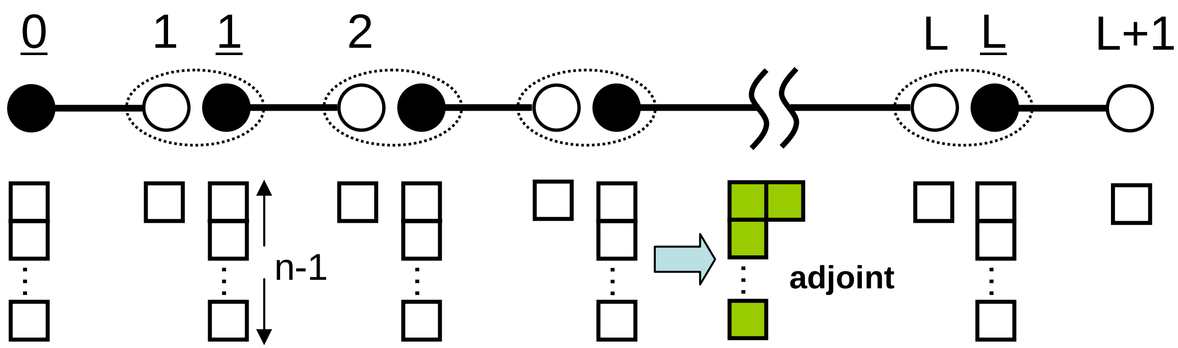
- ***Imhomogeneous AKLT chain (Xu, H.K., Hirano & Korepin, Q. I. P. ('08))***



L → ∞

$$\ln\left[(S_- + 1)(S_+ + 1)\right]$$

- ***SU(n) generalized AKLT (H.K., Hirano & Korepin, J.Phys. A ('08))***



L → ∞

$$2\ln n$$

- ***Periodic boundary cases (Hirano & H.K.): Similar results (L → ∞).***

General property of RDM in VBS (conjecture)

- RDM is a projection operator to the space spanned by edge states.
- Edge state : degenerate ground states of the 'block Hamiltonian'

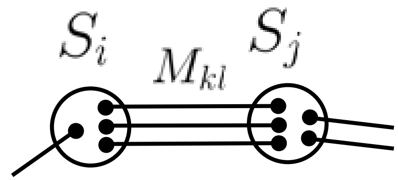
> ***Conjecture***: EE ≦ log[# of degenerate g.s. for the 'block Hamiltonian']

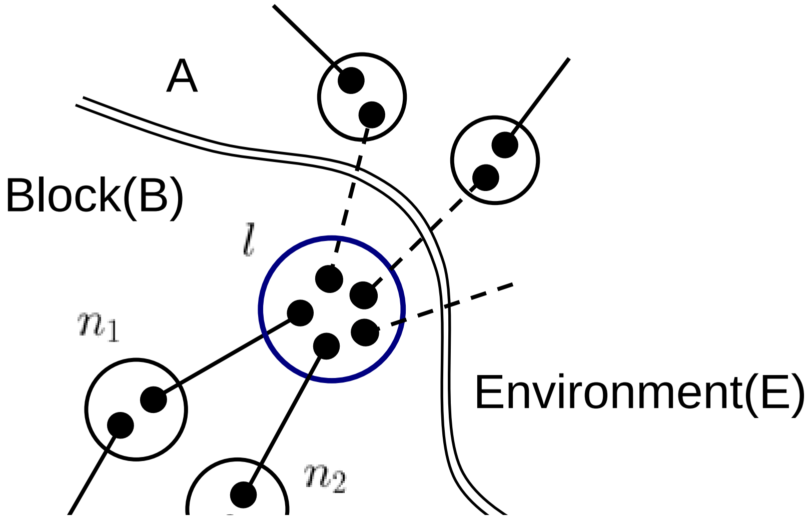
G.S. degeneracy of VBS on a subgraph of arb. graph

- VBS on an arbitrary graph (Kirillov & Korepin, Leningrad. Math. J, ('90))

Condition on unique g.s.

$$2S_l = \sum_k M_{kl}, \qquad \forall\, l.$$





- Block Hamiltonian

$$H_b = \sum_{\langle kl\rangle \in B} H(k,l), \qquad k \in B, \quad l \in B.$$

- Counting formula for gs deg. (H.K ('08))

$$deg. = \prod_{l \in \partial B} \left[\left(\sum_{k \in \partial E} M_{kl} \right) + 1 \right]$$

ex) deg. =4 $\quad (a_l^\dagger)^3|\mathrm{VBS}_{\mathrm{N_b}}\rangle,\ (a_l^\dagger)^2 b_l^\dagger|\mathrm{VBS}_{\mathrm{N_b}}\rangle,\ a_l^\dagger(b_l^\dagger)^2|\mathrm{VBS}_{\mathrm{N_b}}\rangle,\ (b_l^\dagger)^3|\mathrm{VBS}_{\mathrm{N_b}}\rangle$

Conjecture: log[deg.] gives an upper bound on EE (Korepin & Fan ('08)).

Summary of the first topic

1. ***Exact expression for EE in generic VBS states in 1D***
2. ***Relation between EE and correlation functions***
3. ***EE=log[# of edge states]***

	EE	Boundary spins	Degrees of freedom
1	$2\log_2 2$	Two spin-1/2's	$(2\cdot 1/2+1)^2=4$
S	$2\log_2(S+1)$	Two spin-S/2's	$(2\cdot S/2+1)^2=(S+1)^2$

4. ***RDM is exactly spanned by the edge states.***
5. ***VBS on an arb. graph and its application.***

For details, please see

1. H. K., T. Hirano & Y. Hatsugai, Phys. Rrev. B 76, 012401 (2007).
2. H. K., T. Hirano & V. E. Korepin, J. Phys. A 41, 135304 (2008).
3. Y. Xu, H. K., T. Hirano & V. E. Korepin, J. Stat. Phys. 133, 347 (2008).
4. Y. Xu, H. K., T. Hirano & V. E. Korepin, Q. Info. Proc. 7, 153 (2008).

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3.Entanglement in the Calogero-Sutherland (CS) Model

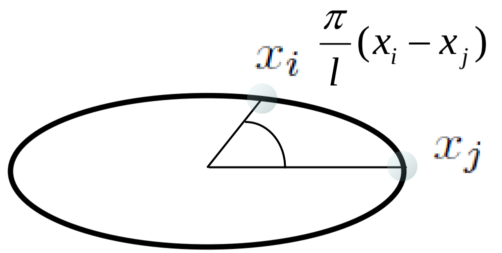
a) CS model and Fractional exclusion statistics

b) L-particle RDM & EE based on particle partitioning

Related paper: 1. H. K. and Y. Hatsuda, J. Phys. A **40** (2007).

1. Introduction to the Calogero-Sutherland Model

$$H_{CS} = -\sum_{j=1}^{N} \frac{1}{2}\frac{\partial^2}{\partial x_j^2} + \sum_{i<j} \frac{\beta(\beta-1)(\frac{\pi}{l})^2}{\sin^2(\frac{\pi}{l}(x_i - x_j))}$$



- **Ground state wave function (Jastrow type):**

$$\psi_0(z_1.z_2,...,z_N) = \frac{1}{\sqrt{N!}}\left(\prod_{j=1}^{N} z_j\right)^{-\beta\frac{N-1}{2}} \prod_{i<j}(z_i - z_j)^{\beta}$$

New coordinate:

$$\left(z_j = \exp\left(\frac{2\pi i}{l}x_j\right)\right)$$

- **Excited state wave function:**

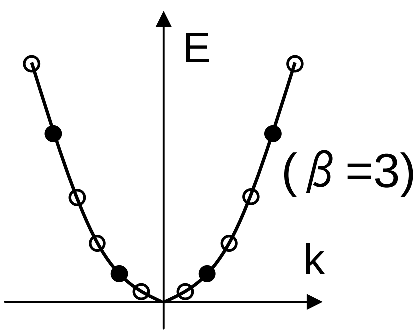
$$\psi_\lambda(z_1,z_2,...,z_N) = P_\lambda(z_1,z_2,...,z_N;\beta)\psi_0(z_1.z_2,...,z_N)$$

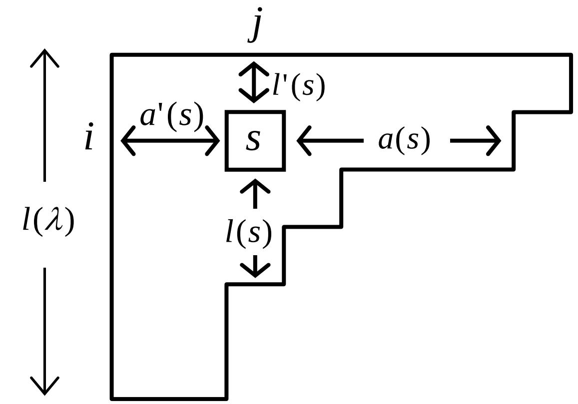
$P_\lambda(z_1,z_2,...,z_N;\beta)$: Jack symmetric polynomial (λ : partition)

- **Eigen-energy and quasi-momentum**

$$E_\lambda = \frac{1}{2}\left(\frac{2\pi}{l}\right)^2 \sum_{i=1}^{N} k_i^2(\lambda)$$

$$k_i(\lambda) = \lambda_i + \beta\left(\frac{N+1}{2} - i\right)$$





Like a free particle with exclusion constraint!

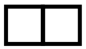
- **Jack Polynomial**

N=5, β =2

···000101010101000···

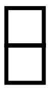
$$P_{(1)} = p_1,$$

···000101010100100···



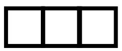
$$P_{(2)} = \frac{1}{1+\beta} p_2 + \frac{\beta}{1+\beta} p_1^2,$$

···000101010100010···



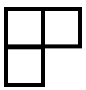
$$P_{(1,1)} = -\frac{1}{2} p_2 + \frac{1}{2} p_1^2,$$

···000101010010100···



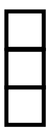
$$P_{(3)} = \frac{2}{(1+\beta)(2+\beta)} p_3 + \frac{3\beta}{(1+\beta)(2+\beta)} p_2 p_1 + \frac{\beta^2}{(1+\beta)(2+\beta)} p_1^3,$$

···000101010100001···



$$P_{(2,1)} = -\frac{1}{1+2\beta} p_3 + \frac{1-\beta}{1+2\beta} p_2 p_1 + \frac{\beta}{1+2\beta} p_1^3,$$

···000101010010010···



$$P_{(1,1,1)} = \frac{1}{3} p_3 - \frac{1}{2} p_2 p_1 + \frac{1}{6} p_1^3.$$

···000101001010100···

Power sum: $p_n = \sum_i z_i^n$

Classical polynomials:

β =0 (monomial symmetric), β =1 (Schur),
β =2 (zonal), β =∞ (elementary) polynomials

2. Fractional exclusion statistics (FES) *(Haldane, PRL ('91))*

- Fermi surface and quasi-holes (β =3)

$-k_F$ $\qquad$ k_F

5-particle g.s.

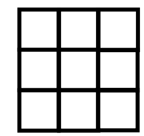
(a) ･･･0 0 0 0 0 (1 0 0) (1 0 0) (1 0 0) (1 0 0) (1 0 0) 0 0 0 0 0 ･･･

Remove one 1.

$$k_i(\lambda) = \lambda_i + \beta(\frac{N+1}{2} - i)$$

3 quasi-holes

(b) ･･･0 0 0 0 0 (1 0 0) $\underbrace{0\ 0\ 0}$ (1 0 0) (1 0 0) (1 0 0) 0 0 0 0 0 ･･･



4-particle g.s.

(c) ･･･0 0 0 0 0 (1 0 0) (1 0 0) (1 0 0) (1 0 0) 0 0 0 0 0 0 0 0 ･･･

Decimation of (N-L) quasi-momenta from the Fermi sea $\Longleftrightarrow$ Creation of β (N-L) quasi-holes in the Fermi sea

# of possible configurations of L particles and β (N-L) quasi-holes:

$$\begin{pmatrix} \beta(N-L)+L \\ L \end{pmatrix}$$

3. Our Main Results

- **Exact L-particle reduced density matrix** (complicated form…)

$$\rho(\overline{w}_1,...,\overline{w}_L;z_1,...,z_L)=\frac{1}{\mathcal{N}(\beta,N)}\frac{1}{\binom{N}{L}}\overline{\Psi_0(w_1,...,w_L)}\Psi_0(z_1,...,z_L)$$

$$\times\sum_{\lambda_1,\lambda_2}\langle P^{(\alpha)}_{\lambda_1}(\{-p_n(z_j)\}),P^{(\alpha)}_{\lambda_2}(\{-p_n(z_j)\})\rangle'_{N-L}\frac{c_{\lambda_1}(\alpha)}{c'_{\lambda_1}(\alpha)}\frac{c_{\lambda_2}(\alpha)}{c'_{\lambda_2}(\alpha)}P_{\lambda_1}(\overline{w}_1,...,\overline{w}_L;\beta)P_{\lambda_2}(z_1,...,z_L;\beta)$$

Original problem → finite-dimensional e.v. problem (dim=# of $\square$)

- **Upper bound value of EE** (*low-density limit:* $(N-L)\to\infty$)

EE: $\mathcal{S}_{N,L}=-\mathrm{Tr}\left[\rho\log\rho\right]$

Upper bound: $\mathcal{S}^{\mathrm{bound}}_{N,L}=\log\binom{\beta(N-L)+L}{L}$

Phys.: # of possible intermediate states

Math.: Duality relation of Jack poly.

Consistent with FQH systems (Zozulya et al, PRB ('07)) $\beta\Longleftrightarrow m$

- **Universal subleading correction**

$$\mathcal{S}^{\mathrm{bound}}_{N,L}-\mathcal{S}_{N,L}\sim L(\log\beta-1+\beta^{-1})$$

Independent of N

4. L-particle Reduced Density Matrix (RDM)

Our setting:

A B

Divide the N-particle system into an L-particle and an (N-L)-partice blocks.

⇒***particle partitioning*** (Another way: spatial partitioning)

L-particle RDM:

N-particle g.s.

$$\rho(\overline{w}_1, ..., \overline{w}_L; z_1, ..., z_L) = \frac{1}{\mathcal{N}(\beta, N)} \oint \frac{dz_{L+1}}{2\pi i z_{L+1}} \cdots \oint \frac{dz_N}{2\pi i z_N} \overline{\psi_0(w_1, ..., w_L, z_{L+1}, ..., z_N)} \psi_0(z_1, ..., z_L, z_{L+1}, ..., z_N)$$

Entanglement Entropy (EE):

$$\mathcal{S}_{N,L} = -\mathrm{Tr}\left[\rho \log \rho\right]$$

Trace in the subsystem A (L-particle block):

$$\mathrm{Tr}\left[A\right] \equiv \oint \frac{dz_1}{2\pi i z_1} \cdots \oint \frac{dz_L}{2\pi i z_L} A(\overline{z}_1, ..., \overline{z}_L; z_1, ..., z_L)$$

$$\mathrm{Tr}\left[AB\right] \equiv \oint \frac{dw_1}{2\pi i w_1} \cdots \oint \frac{dw_L}{2\pi i w_L} \oint \frac{dz_1}{2\pi i z_1} \cdots \oint \frac{dz_L}{2\pi i z_L} A(\overline{w}_1, ..., \overline{w}_L; z_1, ..., z_L) B(\overline{z}_1, ..., \overline{z}_L; w_1, ..., w_L)$$

RDM in terms of the g.s's of subsystems.

$$\rho(\overline{w}_1,...,\overline{w}_L;z_1,...,z_L)=\frac{1}{\mathcal{N}(\beta,N)}\frac{L!(N-L)!}{N!}\left(\prod_{i=1}^{L}\overline{w}_i z_i\right)^{-\beta\frac{N-L}{2}}\overline{\psi_0(w_1,...,w_L)}\psi_0(z_1,...,z_L)$$

$$\times\oint\frac{dz_{L+1}}{2\pi i z_{L+1}}\cdots\oint\frac{dz_N}{2\pi i z_N}\prod_{i=1}^{L}\prod_{j=L+1}^{N}(1-z_i\overline{z}_j)^\beta(1-\overline{w}_i z_j)^\beta|\psi_0(z_{L+1},...,z_N)|^2.$$

g.s. for A

g.s. for B

=

$$\frac{1}{\mathcal{N}(\beta,N)}\frac{1}{\binom{N}{L}}\overline{\Psi_0(w_1,...,w_L)}\Psi_0(z_1,...,z_L)\boxed{\left\langle\prod_{i=1}^{L}\prod_{j=L+1}^{N}(1-\overline{z}_i z_j)^\beta,\prod_{i=1}^{L}\prod_{j=L+1}^{N}(1-\overline{w}_i z_j)^\beta\right\rangle'_{N-L}}$$

Scalar product on the ring of symmetric polynomials:

$$\langle f,g\rangle'_N=\oint\frac{dz_1}{2\pi i z_1}\cdots\oint\frac{dz_N}{2\pi i z_N}\overline{f(z_1,z_2,...,z_N)}g(z_1,z_2,...,z_N)|\psi_0(z_1,z_2,...,z_N)|^2$$

Orthogonality relation for the Jack polynomials:

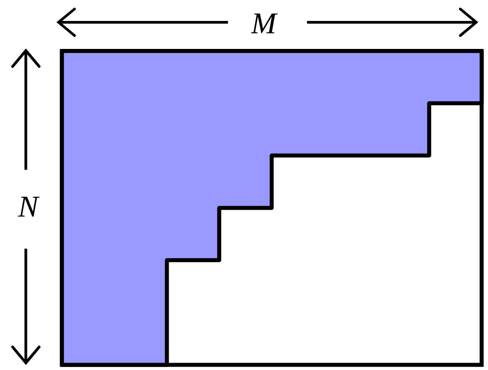
$$\langle P_\lambda,P_\mu\rangle'_N=\delta_{\lambda,\mu}\mathcal{N}(\beta,N)\prod_{s\in\lambda}\frac{a(s)+\beta l(s)+1}{a(s)+\beta l(s)+\beta}\prod_{s\in\lambda}\frac{\beta N+a'(s)-\beta l'(s)}{\beta N+a'(s)+1-\beta(l'(s)+1)}$$

Duality relation of Jack symmetric polynomial:

transpose of λ

$$\prod_{i=1}^{N}\prod_{j=1}^{M}(1+x_iy_j)=\sum_{\lambda}P_{\lambda}(x_1,x_2,...,x_N;\beta)P_{\lambda'}(y_1,y_2,...,y_M;1/\beta)$$

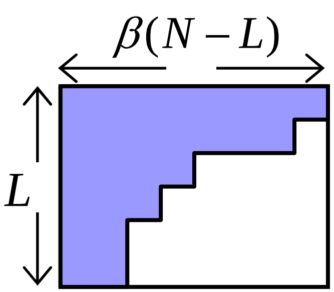
Allowed partitions in the sum:



Replica-like expansion:

$$\prod_{i=1}^{L}\prod_{j=L+1}^{N}(1-\overline{z}_iz_j)^{\beta}=\sum_{\lambda}P_{\lambda}(\overline{z}_1,...,\overline{z}_L;\beta)P_{\lambda'}(\overbrace{-z_{L+1},...,-z_{L+1}}^{\beta},...,\overbrace{-z_N,...,-z_N}^{\beta};1/\beta)$$

\# of possible partitions: $\begin{pmatrix}\beta(N-L)+L\\ L\end{pmatrix}$



Using the **involution** $\omega_\alpha(p_n) = -(-1)^n \alpha p_n$, we can rewrite $P_{\lambda'}$ as

$$P_{\lambda'}(\overbrace{-z_{L+1}, ..., -z_{L+1}}^{\beta}, ..., \overbrace{-z_N, ..., -z_N}^{\beta}; 1/\beta) = \frac{c_\lambda(\alpha)}{c'_\lambda(\alpha)} P_\lambda^{(\alpha)}(\{-p_n(z_j)\})$$

Note: **Power sum:** $p_n = \sum_i z_i^n$ (MacDonald's textbook)

$$P_\lambda^{(\alpha)}(\{+p_n(z_j)\}) = P_\lambda(z_{L+1}, z_{L+2}, ..., z_N; \beta)$$

$$P_\lambda^{(\alpha)}(\{-p_n(z_j)\}) \neq P_\lambda(z_{L+1}, z_{L+2}, ..., z_N; \beta)$$

can be expanded by P_μ *with* $|\mu| = |\lambda|$

Exact (but formal) expression for RDM:

$$\rho(\overline{w}_1, ..., \overline{w}_L; z_1, ..., z_L) = \frac{1}{\mathcal{N}(\beta, N)} \frac{1}{\binom{N}{L}} \overline{\Psi_0(w_1, ..., w_L)} \Psi_0(z_1, ..., z_L)$$

$$\times \sum_{\lambda_1, \lambda_2} \boxed{\langle P_{\lambda_1}^{(\alpha)}(\{-p_n(z_j)\}), P_{\lambda_2}^{(\alpha)}(\{-p_n(z_j)\})\rangle'_{N-L}} \frac{c_{\lambda_1}(\alpha)}{c'_{\lambda_1}(\alpha)} \frac{c_{\lambda_2}(\alpha)}{c'_{\lambda_2}(\alpha)} P_{\lambda_1}(\overline{w}_1, ..., \overline{w}_L; \beta) P_{\lambda_2}(z_1, ..., z_L; \beta)$$

=0, when $|\lambda_1| \neq |\lambda_2|$

⇒***block diagonal structure!***

basis of subsystem A

5. Entanglement Entropy

In (N-L)→∞ limit, a great simplification occurs.

Asymptotic orthogonality:

$$\lim_{N-L\to\infty}\langle P^{(\alpha)}_{\lambda_1}(\{-p_n(\zeta_j)\}), P^{(\alpha)}_{\lambda_2}(\{-p_n(\zeta_j)\})\rangle'_{N-L} = \delta_{\lambda_1\lambda_2}\mathcal{N}(\beta, N-L)\frac{c'_{\lambda_1}(\alpha)}{c_{\lambda_1}(\alpha)}$$

From Macdonald's formulae, e.g. $P^{(\alpha)}_{\lambda}(\{p_n\}) = c_\lambda(\alpha)^{-1}\sum_{\rho}\theta^{\lambda}_{\rho}(\alpha)p_\rho,$

RDM in this limit:

$$\rho(\overline{w}_1,...,\overline{w}_L;z_1,...,z_L) \sim \sum_{\lambda} D_\lambda \tilde{P}_\lambda(\overline{w}_1,...,\overline{w}_L;\beta)\tilde{P}_\lambda(z_1,...,z_L;\beta)\overline{\Psi_0(\{w_j\})}\Psi_0(\{z_j\})$$

$$D_\lambda = \frac{1}{\binom{N\beta}{L\beta}}\prod_{s\in\lambda}\frac{\beta L + a'(s) - \beta l'(s)}{\beta L + a'(s) + 1 - \beta(l'(s)+1)}$$

EE: $\mathcal{S}_{N,L} = -\sum_{\lambda} D_\lambda \log D_\lambda$ # of allowed partitions: $\begin{pmatrix}\beta(N-L)+L \\ L\end{pmatrix}$

Upper bound of EE from a variational argument:

$$\mathcal{S}_{N,L}^{\text{bound}} = \log \binom{\beta(N-L)+L}{L}$$

Consistent with FES!

Upper bound ⇔ L-particle block is ***maximally entangled*** *with the other.*

• ***Universal subleading correction:***

D_λ *depends on the shape of the Young tableau* λ *.*

Using the method developed by Lesage, Pasquier & Serban (NPB ('95))

$$D_\lambda = \frac{\beta^L L!(\beta(N-L))!}{(\beta N)!} \prod_{j=1}^{L} \frac{\Gamma(\lambda_j + \beta(L-j+1))}{\Gamma(\lambda_j + \beta(L-j)+1)}.$$

Scaled variables: $t_j = \lambda_j / N$ $\qquad \Gamma(x+1) \overset{x\to\infty}{\sim} \sqrt{2\pi x}(x/e)^x$

$$\mathcal{S}_{N,L}^{\text{bound}} - \mathcal{S}_{N,L} \sim L(\log\beta - 1 + \beta^{-1})$$

Independent of # of total particles (N)

Summary

1. **Exact expression for RDM**
2. **Upper bound on EE** $\mathcal{S}_{N,L}^{\mathrm{bound}} = \log \binom{\beta(N-L)+L}{L}$

Phys.: # of possibe intermediate states with exclusion constraint.

Math.: Duality relations of Jack symmetric polynomials.

3. **Universal subleading correction**

$$\mathcal{S}_{N,L}^{\mathrm{bound}} - \mathcal{S}_{N,L} \sim L(\log \beta - 1 + \beta^{-1})$$

Independent of the number of total particles (N)

Future Directions

1. **EE based on spatial partitioning**

Fredholm determinant, Dyson's brownian motion of Circular Unitary Ensemble

2. **Applications to lattice integrable models**

Haldane-Shastry, long-range supersymmetric t-J model, etc.