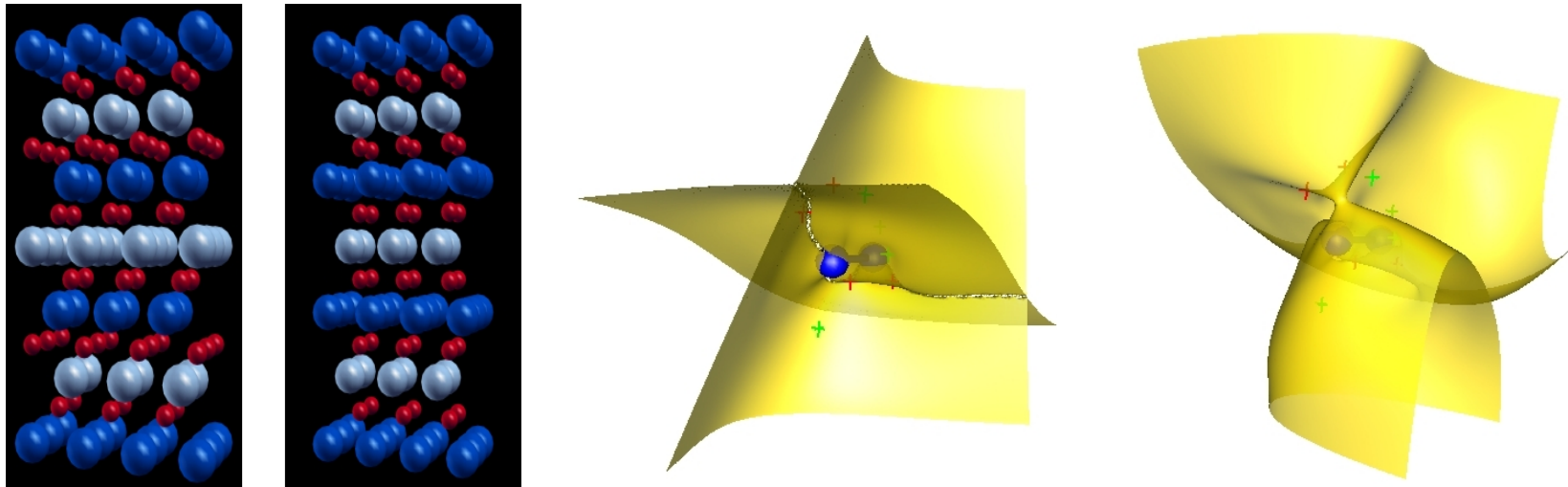


Quantum Monte Carlo: introduction



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Method, in a nutshell

Project out the ground state → imaginary time Schrodinger eq. (Fermi 1933)

$$\psi(\mathbf{R}, t) = \exp(-tH) \psi_T(\mathbf{R}) \quad \rightarrow \quad \psi(\mathbf{R}, t \rightarrow \infty) \propto \phi_0(\mathbf{R})$$

projector in parameter t trial wave function ground state of given symm.

H → interacting quantum particles, eg, electrons + ions

$\mathbf{R} = (r_1, r_2, \dots, r_N)$ → 3N-dim. continuous space

Projection/evolution equation → Euclidian/imaginary time Sch. eq.

$$-\partial_t \psi(\mathbf{R}, t) = H \psi(\mathbf{R}, t)$$



$$\psi(\mathbf{R}, t + \tau) = \int G(\mathbf{R}, \mathbf{R}', \tau) \psi(\mathbf{R}', t) d\mathbf{R}'$$

Green's function $G(\mathbf{R}, \mathbf{R}', \tau) = \langle \mathbf{R} | \exp(-\tau H) | \mathbf{R}' \rangle$ → transition probability

Stochastic method for solving the evolution

$$\Psi(\mathbf{R}, t + \tau) = \int G(\mathbf{R}, \mathbf{R}', \tau) \Psi(\mathbf{R}', t) d\mathbf{R}'$$

Map it onto an equivalent stochastic process:

Value of the wavefunction \leftrightarrow density of sampling points in $3N$ -space

$$\Psi(\mathbf{R}, t) = \text{dens} \left[\sum_i^{\text{walkers}} \delta(\mathbf{R} - \mathbf{R}_i(t)) \right] + \epsilon_{\text{statistical}}$$

sampling points \rightarrow “walkers” \rightarrow eigenstates of position operator

Solution: take short-time approx. to $G(\mathbf{R}, \mathbf{R}', \tau)$ and iterate

Essentially: Feynman path integrals in Euclidean time

Toy model: 1D harmonic oscillator

$$H = T + V(x)$$

Propagator

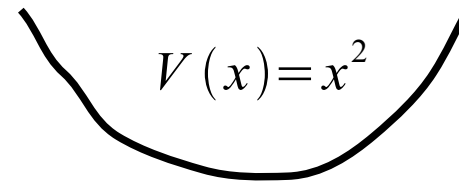
$$G(x, x', \tau)$$



$$C e^{-(x-x')^2/2\tau} \cdot e^{-(V(x)-E_T)\tau}$$

diffusion

weight

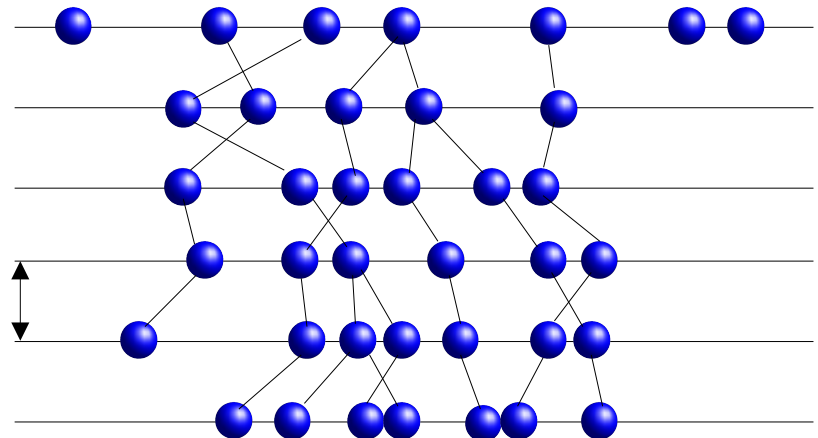


$$\Psi_{init}(x)$$

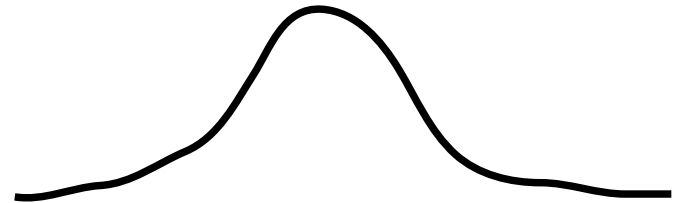


$t \downarrow$

τ



$$\Psi_{ground}(x)$$



Sign problem: fermionic wave functions are both + and -

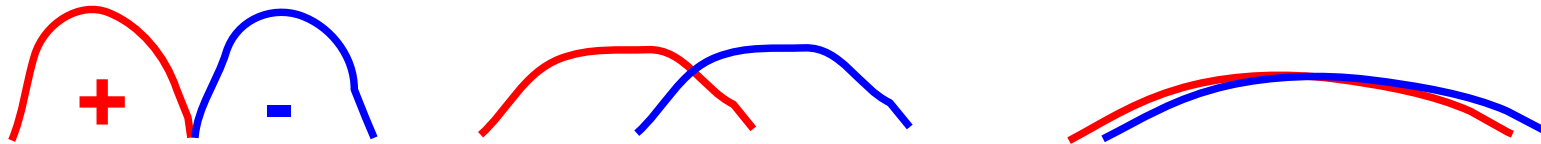
Naïve approach: decompose to + and -

$$\Psi_T(\mathbf{R}) = \Psi_T^+(\mathbf{R}) - \Psi_T^-(\mathbf{R})$$

$$-\partial_t \Psi^+(\mathbf{R}, t) = H \Psi^+(\mathbf{R}, t)$$

$$-\partial_t \Psi^-(\mathbf{R}, t) = H \Psi^-(\mathbf{R}, t)$$

However, + and - components are independent (linearity of Sch. eq.) → both components converge to the lowest energy solution → bosonic !



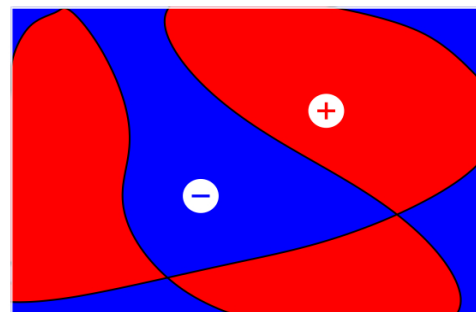
$$\lim_{t \rightarrow \infty} \Psi^+(\mathbf{R}, t) - \lim_{t \rightarrow \infty} \Psi^-(\mathbf{R}, t) \propto \exp[-(E_{Fermionic} - E_{Bosonic})t]$$

Fermionic "signal" decays exponentially quickly into a bosonic "noise"

Solution: impose a constraint \rightarrow fixed-node approximation diffusion Monte Carlo (FNDMC)

Fixed-node (FN) approximation:

$$\text{sign}[\phi(\mathbf{R}, t)] \stackrel{!}{=} \text{sign}[\psi_T(\mathbf{R})]$$



Then the product is nonnegative: $\psi_T(\mathbf{R})\phi(\mathbf{R}, t) = f(\mathbf{R}, t) > 0$

Modify the Schr. eq. accordingly: $f(\mathbf{R}, t + \tau) = \int G^*(\mathbf{R}, \mathbf{R}', \tau) f(\mathbf{R}', t) d\mathbf{R}'$

The projection $f(\mathbf{R}, t \rightarrow \infty) \propto \psi_T(\mathbf{R})\phi_{\text{ground}}(\mathbf{R})$ now depends on

the fermion node: (3N-1)-dim. hypersurface defined as $\phi(r_1, r_2, \dots, r_N) = 0$

Clearly, the node divides the configuration space into + and - domains.

Fermion node toy model: excited state of harmonic oscillator

$$H = T + V(x)$$

Propagator

$$G(x, x', \tau)$$

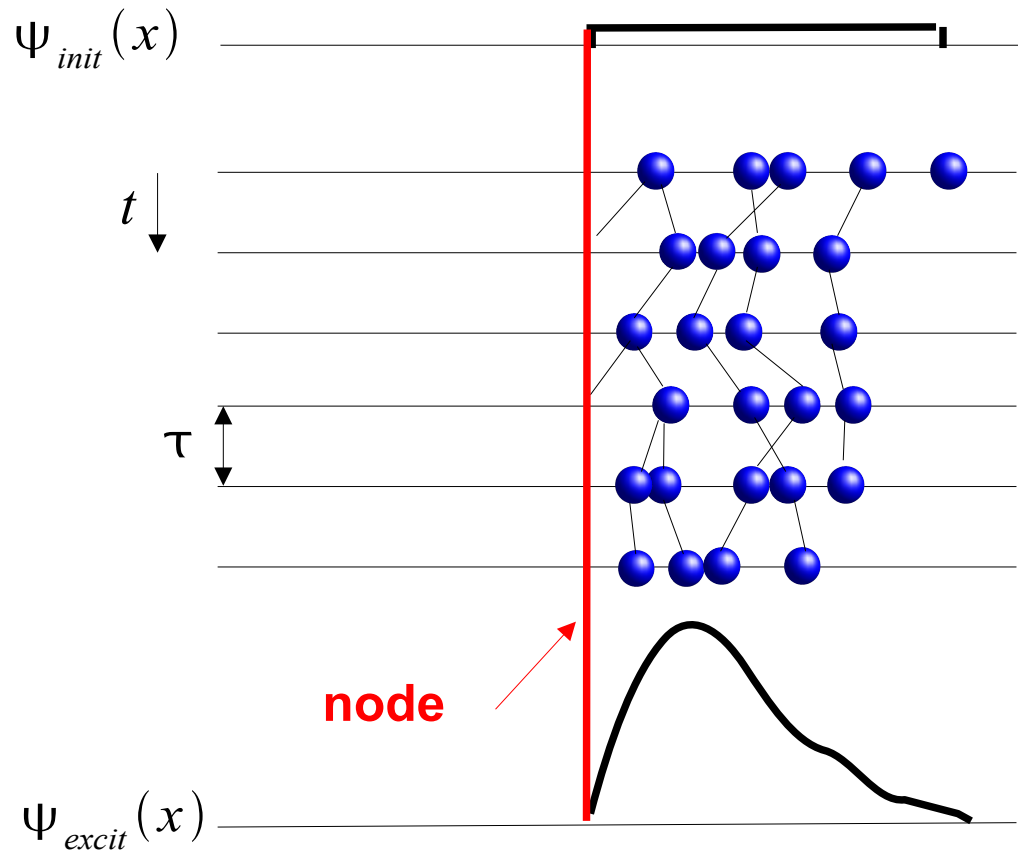
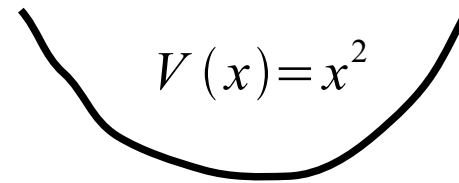


$$C e^{-(x-x')^2/2\tau} \cdot e^{-(V(x)-E_T)\tau}$$

diffusion

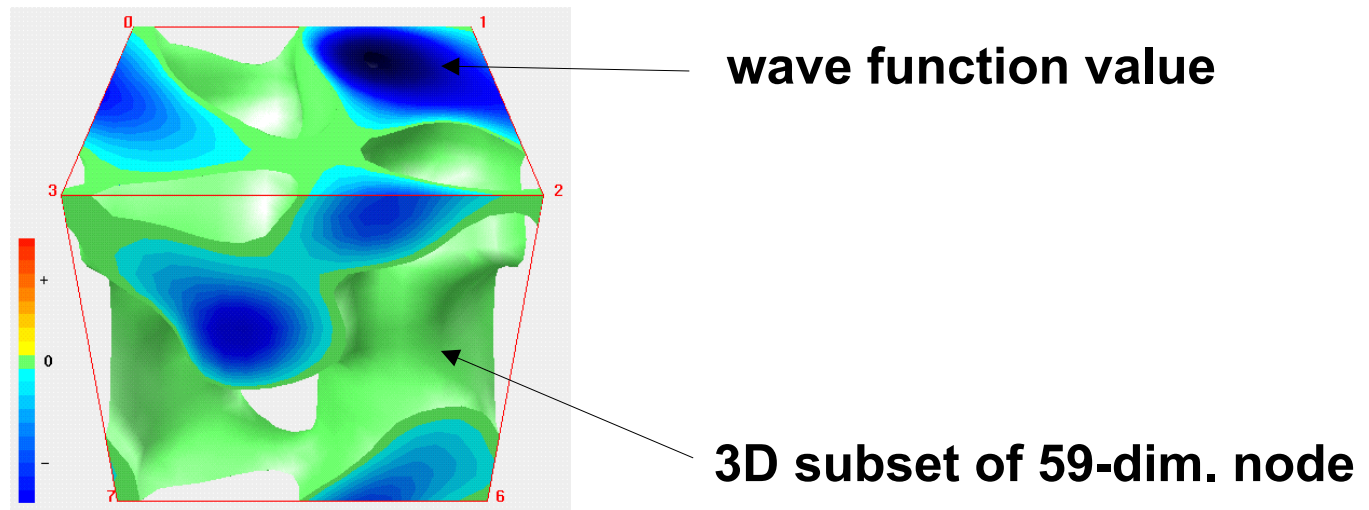
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+ node (boundary cond.)



Fixed-node approximation and fermion nodes

- antisymmetry (nonlocal) replaced by a boundary (local) → boundaries are easy to enforce
- exact node implies recovering exact energy (in polynomial time)



- exact nodes generally unknown, however, **approximate nodes surprisingly accurate (and systematically improvable)**

QMC calculations: basic steps

Hamiltonian: - often valence e- only, using pseudopots/ECPs
- explicit e-e interactions, full many-many body

Trial wave functions:

- correct symmetries
- sampling efficiency
- capture the physics

Commonly used **correlated Slater-Jastrow** type:

$$\psi_{Trial} = \det^{\uparrow}[\{\phi_{\alpha}\}] \det^{\downarrow}[\{\phi_{\beta}\}] \exp[U_{corr}]$$

or

$$\psi_{Trial} = \sum_k c_k \det_k^{\uparrow}[\{\phi_{\alpha}\}] \det_k^{\downarrow}[\{\phi_{\beta}\}] \exp[U_{corr}]$$

QMC calculations: basic steps II

Orbitals $\{\phi_\alpha\}, \{\phi_\beta\}$ from : - Hartree-Fock, post-HF
- Density Functional Theory, hybrid DFT
- possibly CI (natural orbitals), etc

→ QMC interfaced with other codes

Explicit correlations: $U_{corr} = \sum_{i,j} f_{e-e}(r_{ij}) + \sum_{i,I} f_{e-ion}(r_{iI}) + \dots$
- optimized variationally

$$E_{VMC} = \frac{\int \psi_T^2 [H \psi_T / \psi_T] d\mathbf{R}}{\int \psi_T^2 d\mathbf{R}} = \frac{1}{M} \sum_{sample} \frac{H \psi_T(\mathbf{R}_{sample})}{\psi_T(\mathbf{R}_{sample})} + \epsilon_{stat} (1/\sqrt{M})$$

where the samples are distributed as $\psi_T^2(\mathbf{R})$

QMC calculations: basic steps III

Quantities which do not commute with Hamiltonian are more complicated → DMC produces only mixed estimators

$$\langle A \rangle_{DMC} = \langle \psi_T | A | \psi_{DMC} \rangle$$

Correction:

$$\langle A \rangle \approx 2 \langle \psi_T | A | \psi_{DMC} \rangle - \langle \psi_T | A | \psi_T \rangle$$

Methods such as reptation MC sample the square of the wave function but significantly more expensive

QMC calculations of solids

Solids: periodic supercells

- Coulomb potential energy → Ewald sums
- kinetic energy: sampling of k-points of the supercell Brillouin zone → twist averages
- twist average states are not necessarily periodic with the supercell, neither necessarily real (fixed-node can be generalized to fixed-phase, more on the fixed-phase later)
- thermodynamic limit: finite size corrections (for metals this could be a challenge, eg, for a complicated Fermi surface)

How does it work ?

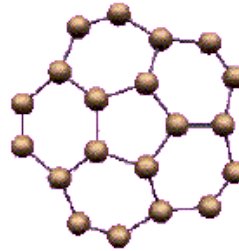
Let us look at a few applications

Some history: what is the lowest energy isomer of C_{20} ???

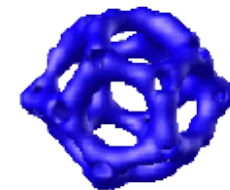
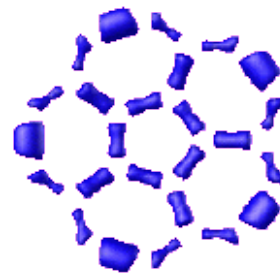
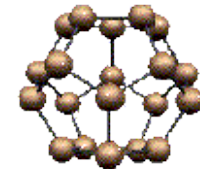
ring



bowl

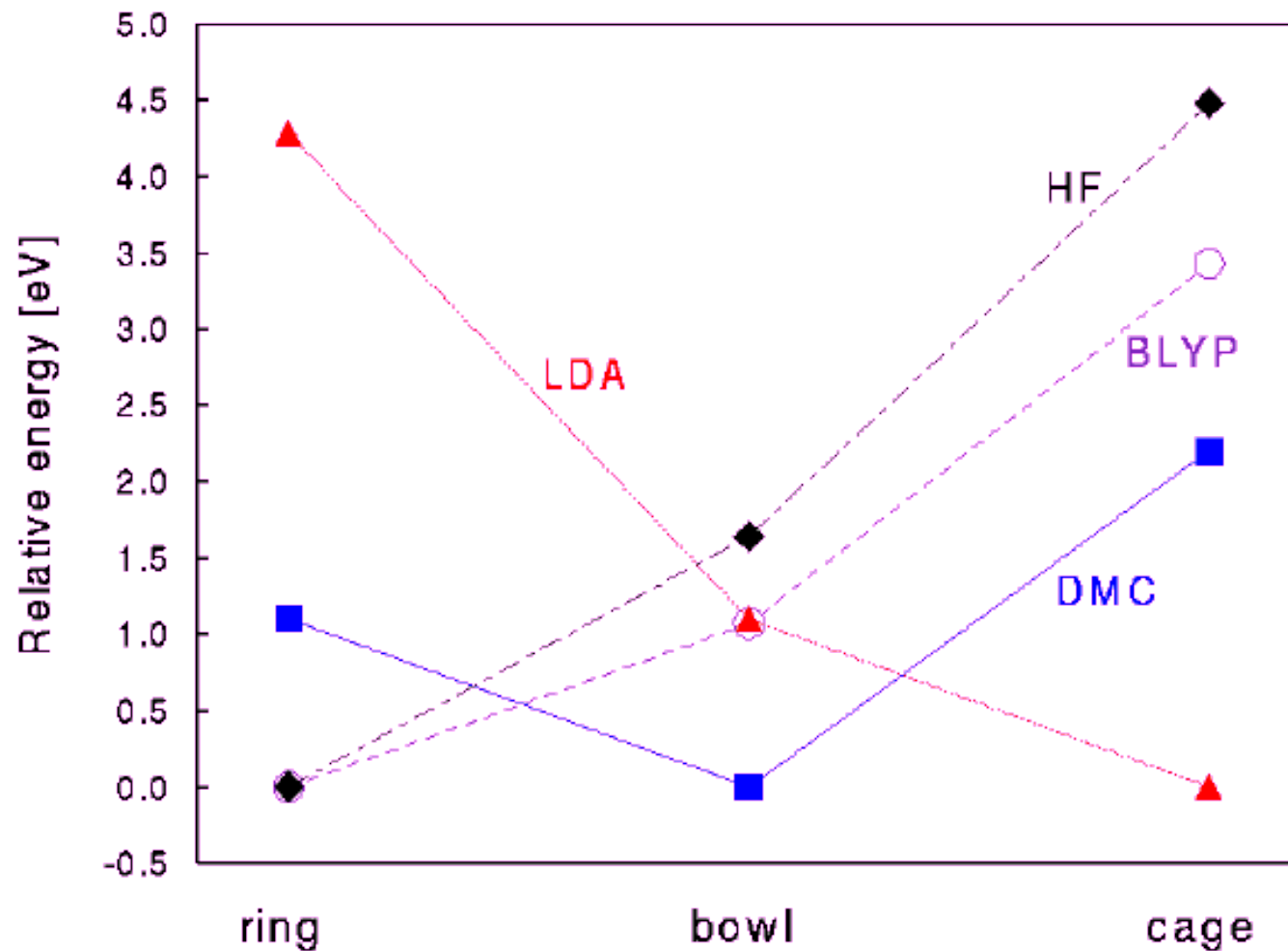


cage



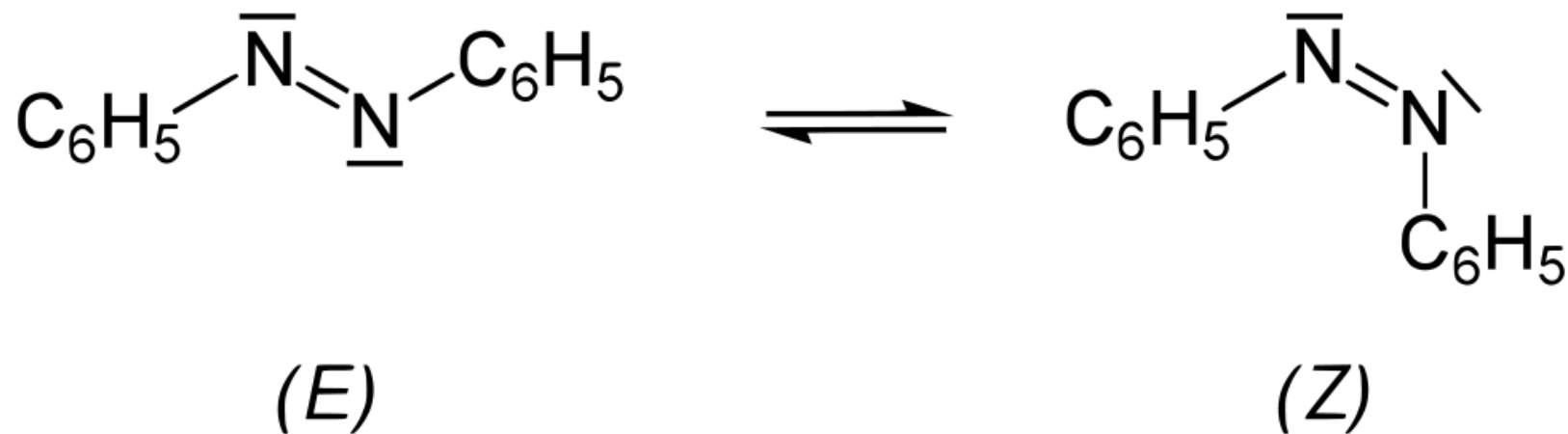
J.C. Grossman, LM, K. Raghavachari, PRL 75, 3870 (1995)

QMC was the first method to predict this (later confirmed by independent methods)

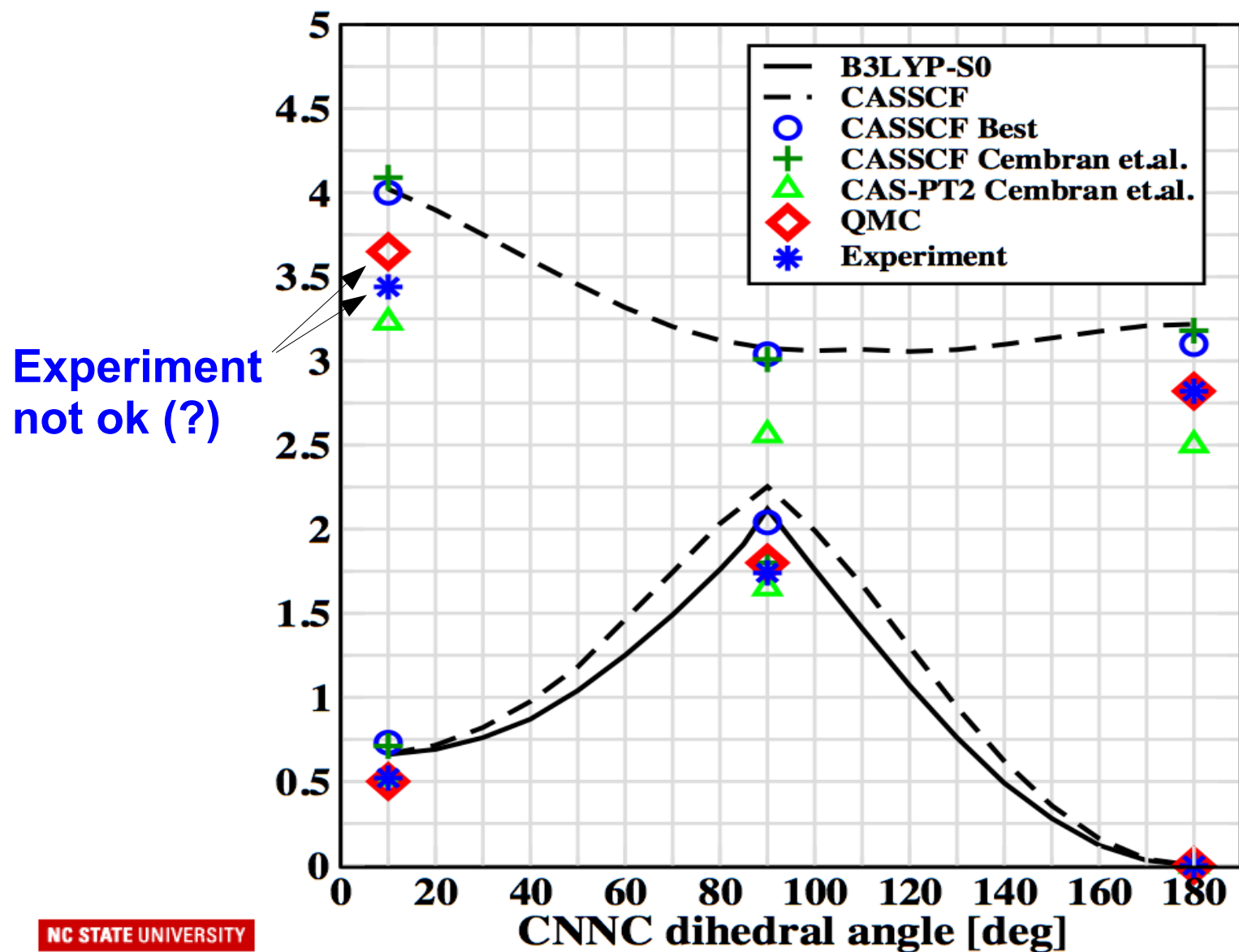


J.C. Grossman, LM, K. Raghavachari, PRL 75, 3870 (1995)

Azobenzene: optically active molecule with photoisomerization



Ground and excited states of azobenzene: ~ 0.04 eV accuracy with FNDMC/multi-det.

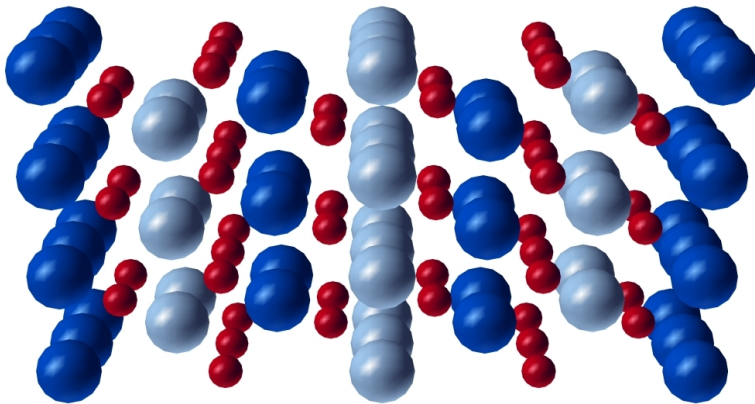


M. Kostolny, R. Derian,
I. Stich, L.M. JCP '10

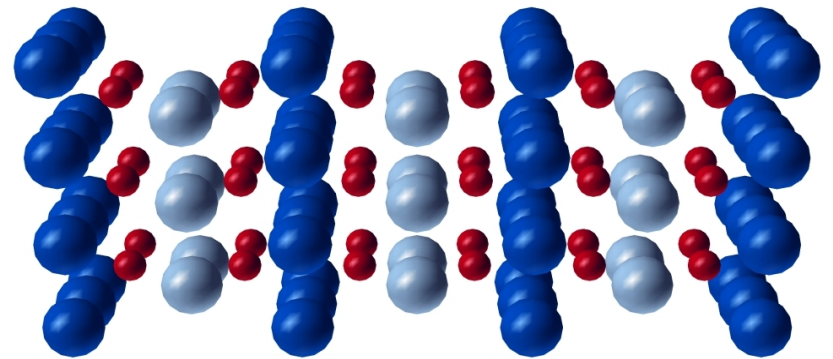
More recent challenge: FeO solid at high pressures

- **large e-e correlations, difficult**: competition of Coulomb, exchange, correlation and crystal-field effects; open d-shell; important **high-pressure physics** (Earth interior, for example)
- mainstream Density Functional Theories (DFT) predict:
 - **metal instead of a large-gap insulator**
 - **wrong equilibrium atomic structure**

B1 (NaCl) AFII (true equil.)



iB8 (NiAs) AF (high press.)



Plain vanilla fixed-node DMC for the FeO solid

- Ne-core, scalar relativistic pseudopotentials on Fe
- 8 supercells (176 valence e-) of FeO in DMC, larger supercells in VMC
- total energy about 4000 eV, trying for accuracy 0.1 eV
- Slater-Jastrow wf $\psi_{Trial} = \det^{\uparrow}[\phi_{\alpha}] \det^{\downarrow}[\phi_{\beta}] \exp[U_{corr}]$
- one-particle orbitals from hybrid DFT (more on that later)

Comparisons of the FeO solid equilibrium parameters

	DFT/PBE	FNDMC	Exp.(FeO _{1-x})
iB8-B1/AFMII [eV]	- 0.2	0.5 (1)	>0
Cohesion [eV]	~ 11	9.7(1)	9.7(2)
a ₀ [Å]	4.28	4.32(1)	4.31- 4.33
K ₀ [GPa]	191	170(10)	140 - 180
Gap [eV]	~ 0 (metal)	2.8(4)	~ 2.4

Gap and excitations in QMC

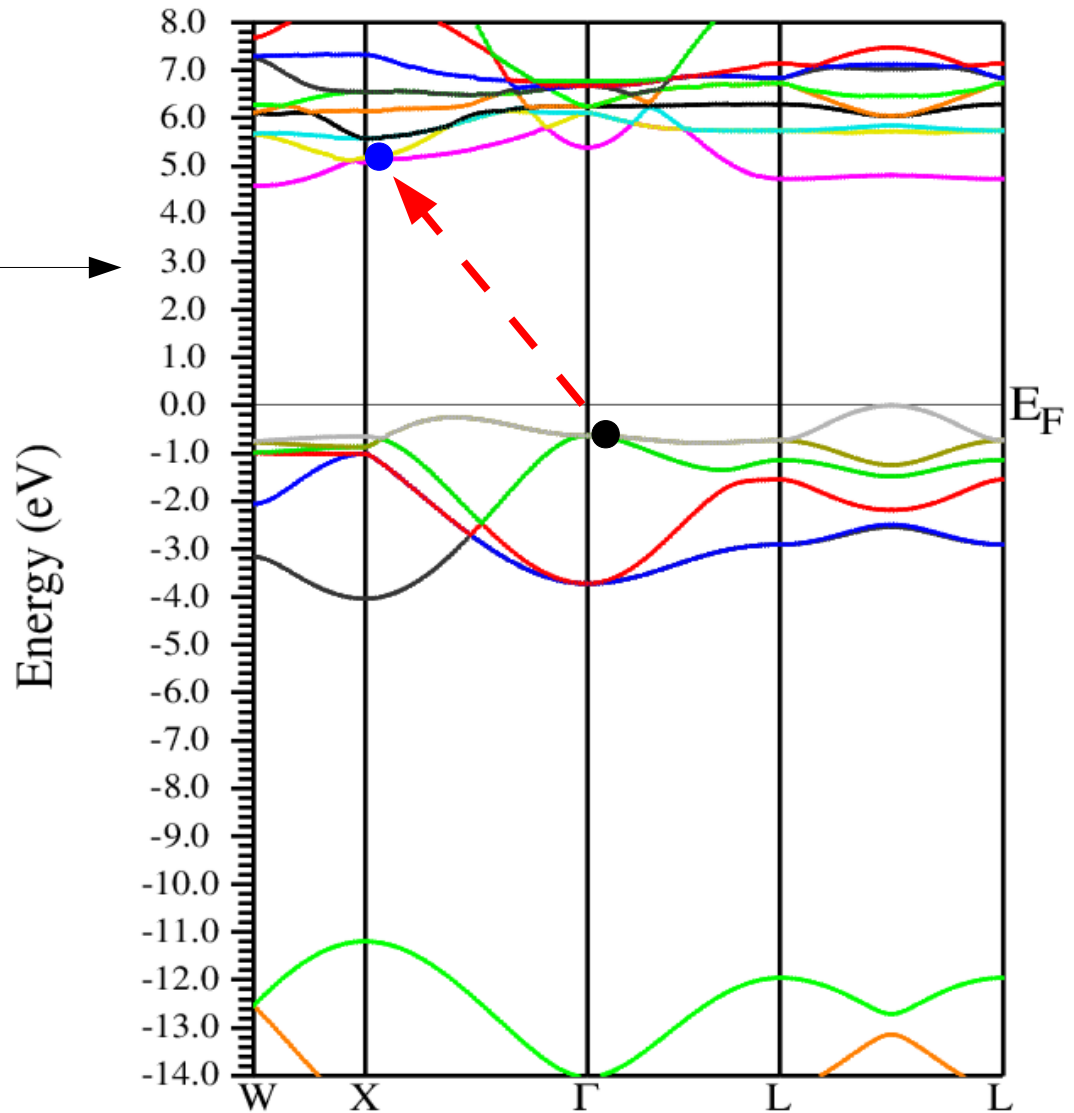
Assume an insulator:

1) Excite a valence state
into a conduction state

$$E_{\text{gap}} \sim E_{\text{excit}} - E_{\text{ground}}$$

- carry out several of these
→ “band structure scan”

2) Add an electron
(technically more
complicated, but doable)



Excitations ? They are not the ground states ...

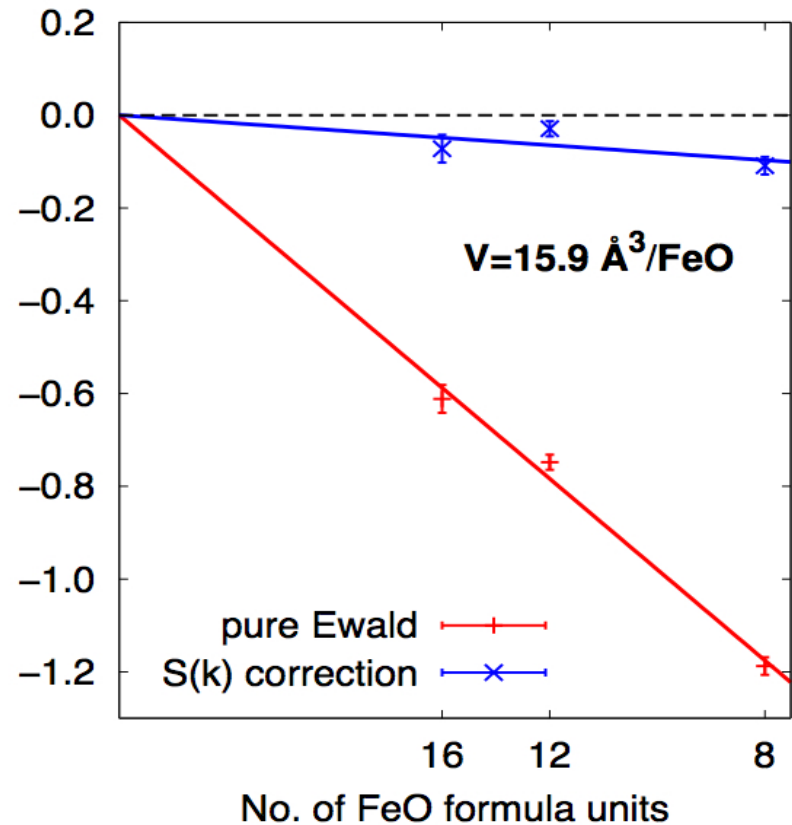
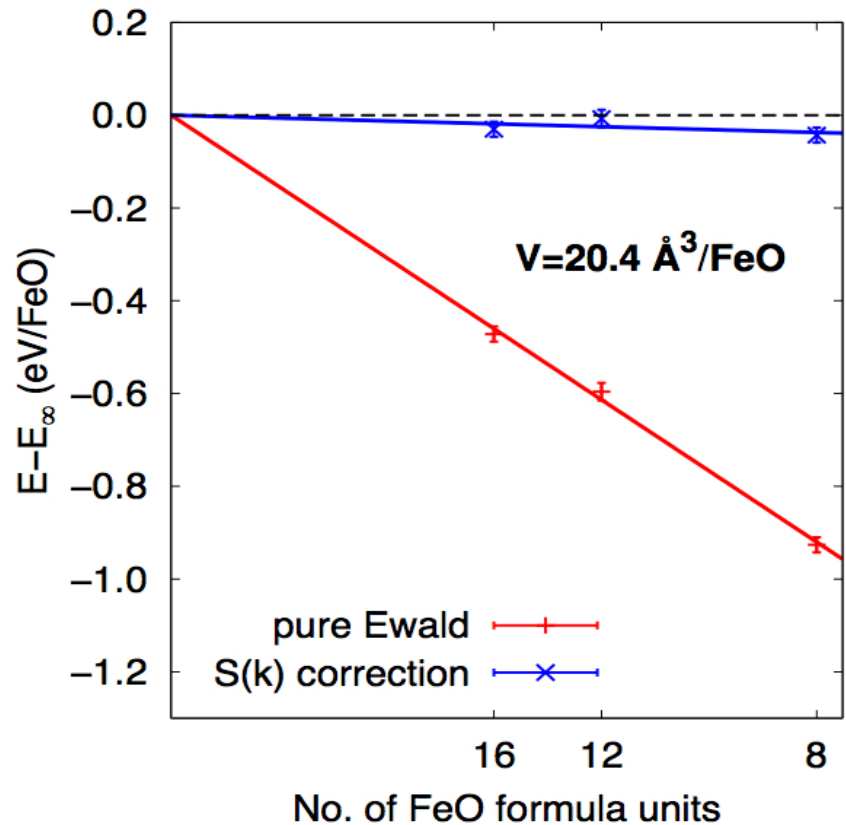
Still, excitations can be calculated (after all, even the fermionic ground state is an excited state!)

Basic possibilities:

- 1) Excitation is of a different symmetry → so, the ground state of the given symmetry**
- 2) Excitation is of the same symmetry → suprisingly, it often works similarly as the ground state calculation.
(How come ? The reason is that the overlap with the ground state is typically much, much smaller than the fixed-node error.)**
- 3) The reasoning in 2) does not apply (it can happen). Then explicitly orthogonalize (that is more complicated but not impossible).**

Finite size corrections → works reasonably well for insulators

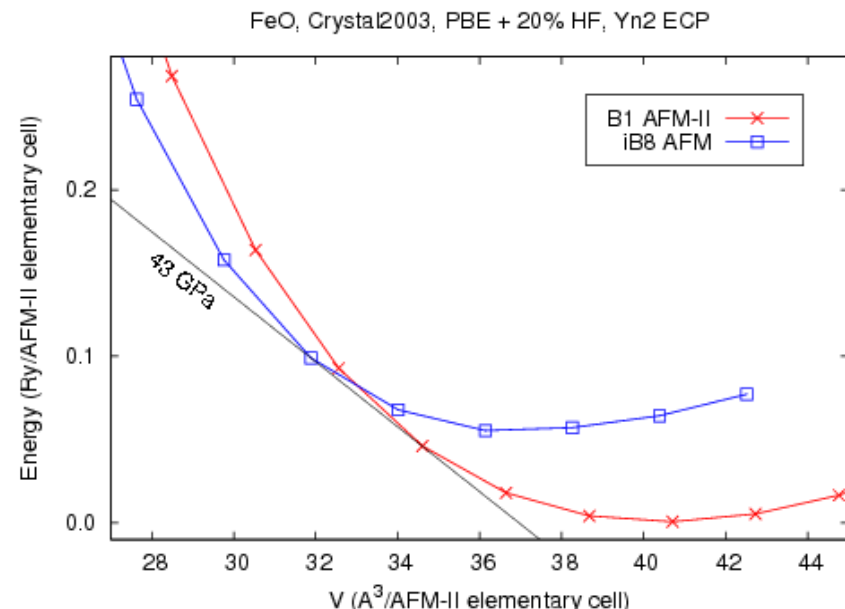
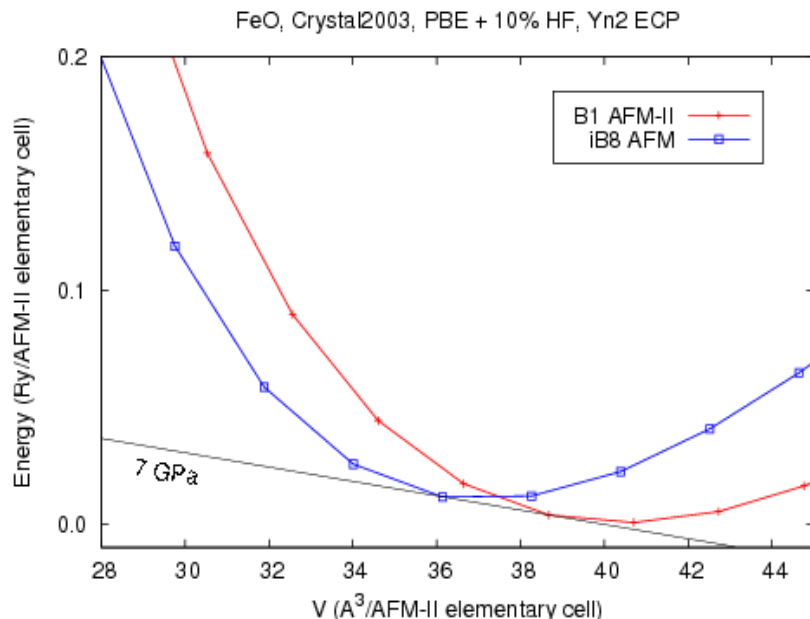
Extrapolation to the thermodynamic limit at two different volumes: $\sim 1/N$
 $S(k)$ includes corrections both for potential and kinetic energies



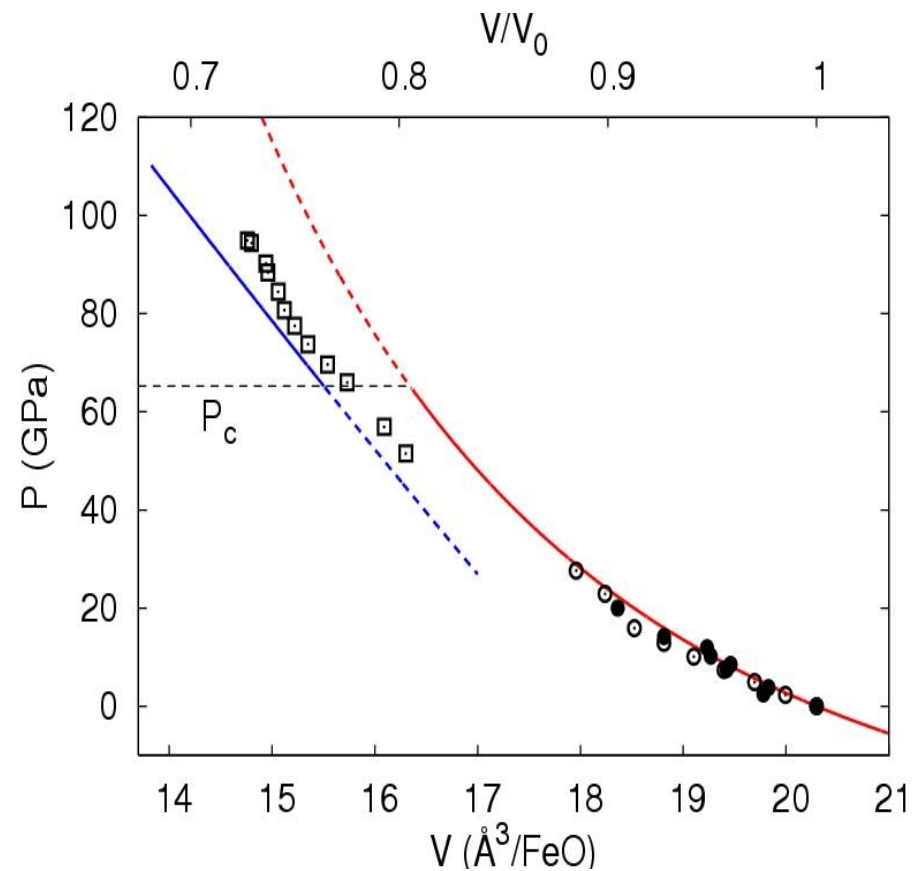
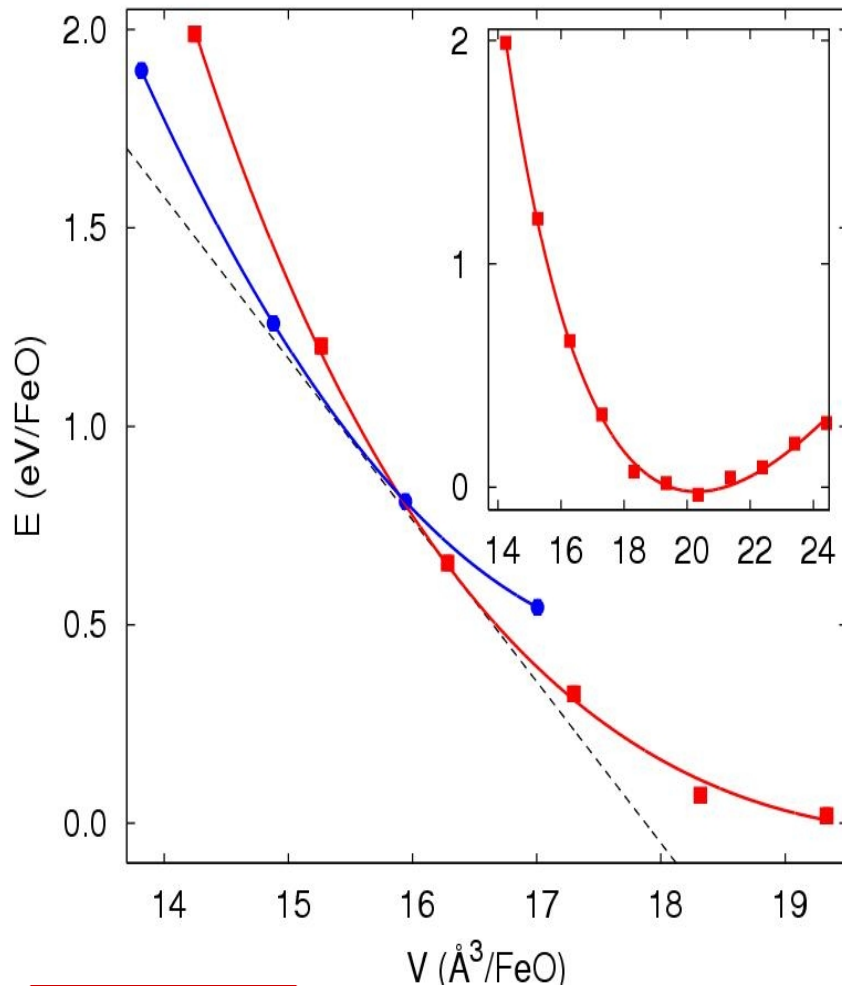
FeO solid at high pressures in DFT: hybrids with weighted exact exchange (or DFT+U)

In order to **reconcile DFT with experiment** one needs to employ mixing of the exact exchange: **transition pressure proportional to mixing ... or to the Hubbard U** (eg, Fang, Terakura, Sawada, Miyazaki, Solovyev, '98)

Complicated to justify the “correct” weight or U, non-variational, ...



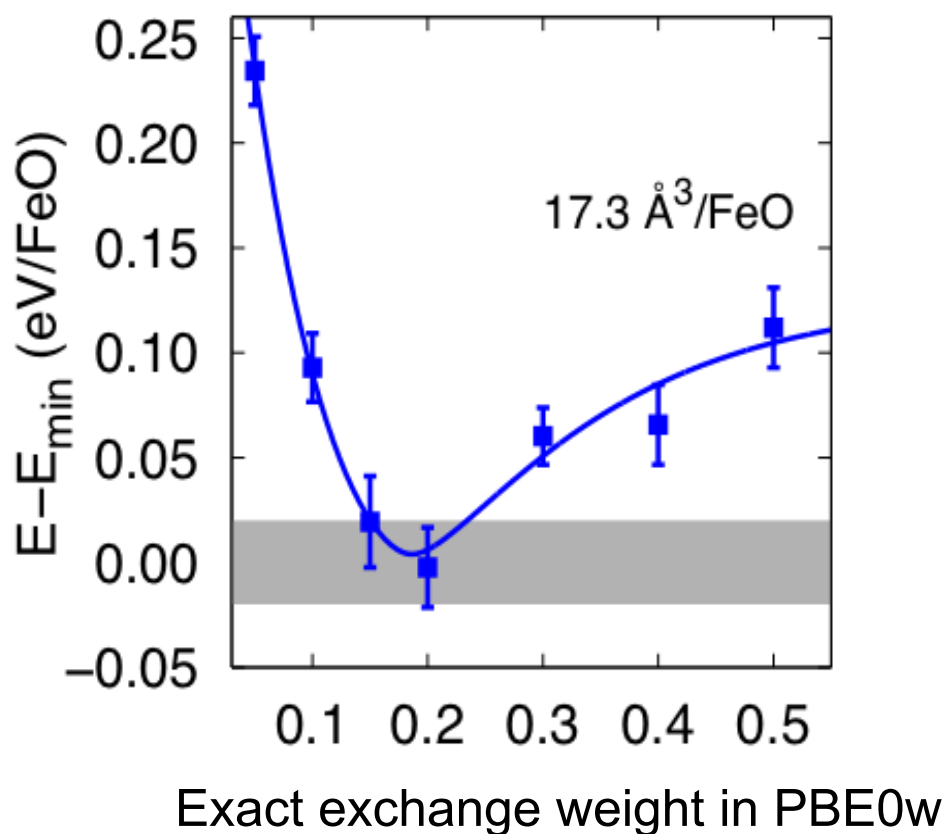
Equation of state of FeO solid at high pressures: QMC shows transition at ~ 65 GPa (Exper. 70-100)



Orbitals: hybrid functional with varying weight of exact/Fock exchange

Weight optimized by FNDMC: clear minimum!

$$\psi_{\text{Trial}} = \det^{\uparrow}[\{\phi_{\alpha}^{\text{PBE0}_w}\}] \det^{\downarrow}[\{\phi_{\beta}^{\text{PBE0}_w}\}] \exp[U_{\text{corr}}]$$



HF weight \rightarrow d-p hybridization: HF “ionic” vs DFT “covalent”

QMC byproduct: construction of optimal effective Hamiltonians (one-body or beyond)

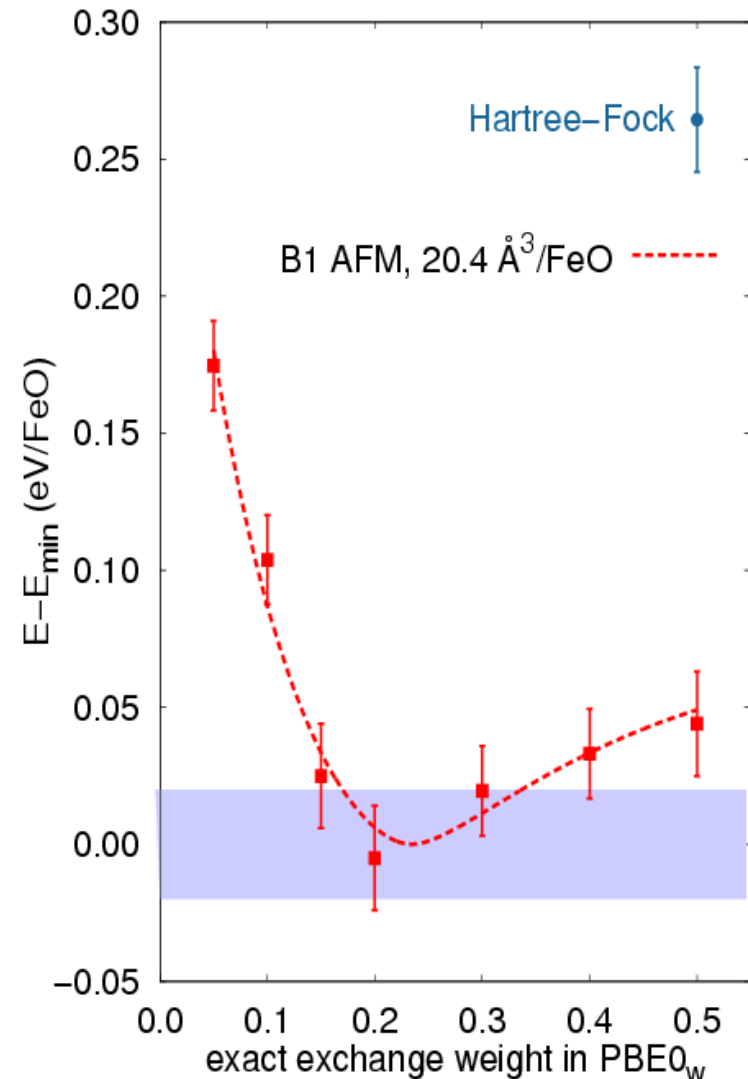
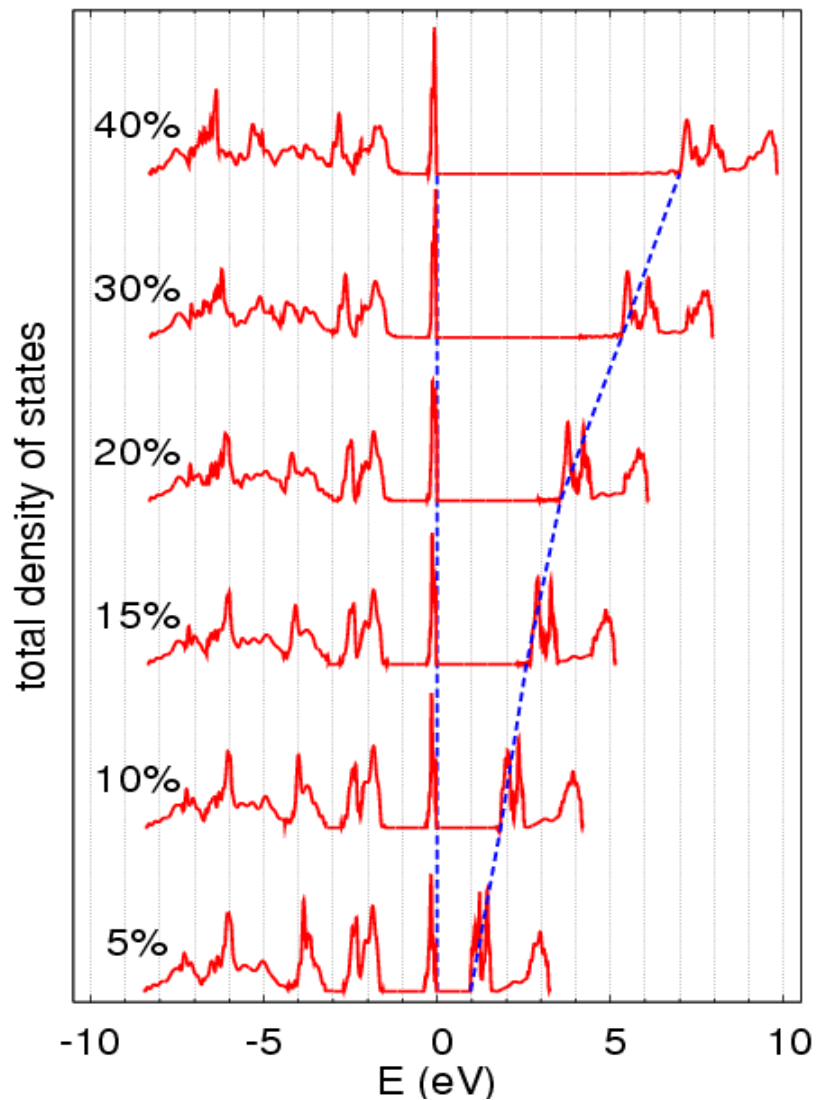
The mixing of exact exchange into the effective one-particle (DFT) Hamiltonian is clearly justified:

- variationally optimized fixed-node DMC energy
- **orbitals beyond Hartree-Fock** → **correlated** (the most appropriate orbitals need not to necessarily minimize the HF energy)
- points out towards a more general idea/tool: **variational space** includes not only the wave function but also **optimal effective Hamiltonian** (possibly more efficient and faster generation of accurate nodes)

FNDMC used as a variational theory →

J. Kolorenc, S. Hu, LM PRB 82 115108 ('10)

Enables to look (“back”) at the **optimal** one-particle picture within the effective Hamiltonian, density of states, gap, etc



Beyond Slater-Jastrow wave functions:

BCS and pfaffians

Why beyond Slater-Jastrow ?

Slater-Jastrow:

$$\psi_{Trial} = \det^{\uparrow}[\phi_{\alpha}] \cdot \det^{\downarrow}[\phi_{\beta}] \exp[U_{corr}]$$



$$node = (node^{\uparrow}) \cdot (node^{\downarrow})$$

Strictly speaking, nodes have such product form only in non-interacting systems → the nodal domains count is **higher than it should be → sometimes this still an excellent approximation while in other cases it does matter**

Possibilities to take unlike spin correlations into account:

- 1) linear combination of determinants (CI)**
- 2) more general antisymmetric forms**

Possible antisymmetric forms (polynomial complexity)

Slater determinant: $\psi_{HF}(1,2,\dots,N) = A \prod_i \phi_i(j) = \det[\phi_i(j)]$
↑
single-particle orbitals

BCS wave function (spin singlet, fixed-number of pairs, in first quantization):

$$\psi_{BCS} = \det[\phi^{\uparrow\downarrow}(i,j)] \quad i, j = 1, \dots, N$$

↑
pair orbital

Pfaffian: (any spin state, antisymmetrized pairs of any spin)

$$\psi_{PF} = A[\phi(1,2)\phi(3,4)\dots] = pf[\phi(i,j)] \quad i, j = 1, \dots, 2N$$

↙ ↘
pair spinorbital

Pfaffian: signed sum of all distinct pair partitions of permutations (Pfaff, Cayley ~ 1850) → polynomial complexity

$$pf[a_{ij}] = \sum_P (-1)^P a_{i_1 j_1} \dots a_{i_{2N} j_{2N}}, \quad i_k < j_k, \quad k = 1, \dots, 2N$$

- determinant is a special case of pfaffian (**pfaffian is more general**)
- pfaffian algebra similar to determinants (minors, etc) → fast evaluation, $O(N^3)$
- ψ_{HF}, ψ_{BCS} special cases of ψ_{PF}

$$\phi(x_i, x_j) = \phi^{\uparrow\downarrow}(r_i, r_j)(\uparrow\downarrow - \downarrow\uparrow) + \chi^{\uparrow\uparrow}(r_i, r_j)(\uparrow\uparrow) + \chi^{\downarrow\downarrow}(r_i, r_j)(\downarrow\downarrow) + \chi^{\uparrow\downarrow}(r_i, r_j)(\uparrow\downarrow + \downarrow\uparrow)$$



Pfaffian wavefunctions with both singlet and triplet pairs (beyond BCS!) → all spin states treated consistently: simple, elegant

$$\Psi_{PF} = pf \begin{bmatrix} \chi^{\uparrow\uparrow} & \phi^{\uparrow\downarrow} & \psi^{\uparrow} \\ -\phi^{\uparrow\downarrow T} & \chi^{\downarrow\downarrow} & \psi^{\downarrow} \\ -\psi^{\uparrow T} & -\psi^{\downarrow T} & 0 \end{bmatrix} \times \exp[U_{corr}]$$

- pairing orbitals (geminals) expanded in one-particle basis

$$\begin{aligned} \phi(i, j) &= \sum_{\alpha \geq \beta} a_{\alpha\beta} [h_{\alpha}(i)h_{\beta}(j) + h_{\beta}(i)h_{\alpha}(j)] \\ \chi(i, j) &= \sum_{\alpha > \beta} b_{\alpha\beta} [h_{\alpha}(i)h_{\beta}(j) - h_{\beta}(i)h_{\alpha}(j)] \end{aligned}$$

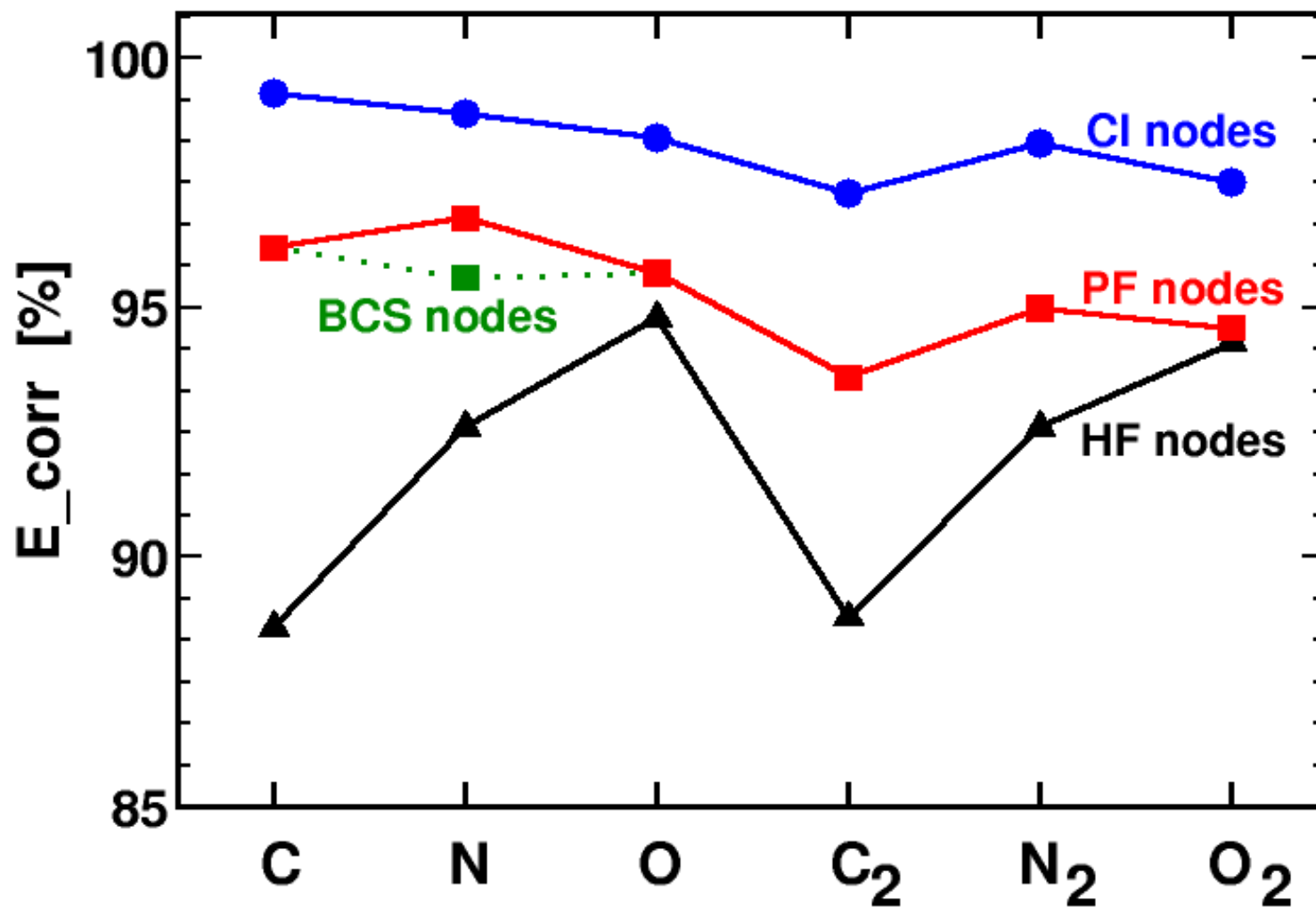
- unpaired

$$\psi(i) = \sum_{\alpha} c_{\alpha} h_{\alpha}(i)$$

BCS wf. for 2N-particle singlet is a special case: $\Psi_{BCS} = \det[\phi^{\uparrow\downarrow}]$

(M. Bajdich et al, PRL '06; PRB '08)

DMC correlation energies of atoms, dimers
Pfaffians: more accurate and **systematic** than HF
while **scalable** (unlike CI)



Expansions in many pfaffians for first row atoms: FNDMC ~ 98 % of correlation with **a few** pfaffians

Table of correlation energies [%] recovered: MPF vs CI nodes

n = # of pfs/dets

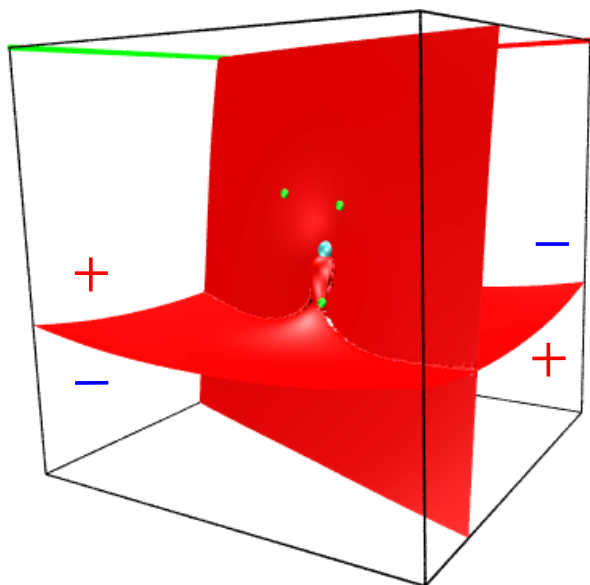
WF	n	C	n	N	n	O
DMC/MPF	3	98.9	5	98.4	11	97.2
DMC/CI	98	99.3	85	98.9	136	98.4

- further generalizations: pairing with backflow coordinates, independent pairs, etc (M. Bajdich et al, PRL 96, 130201 (2006))

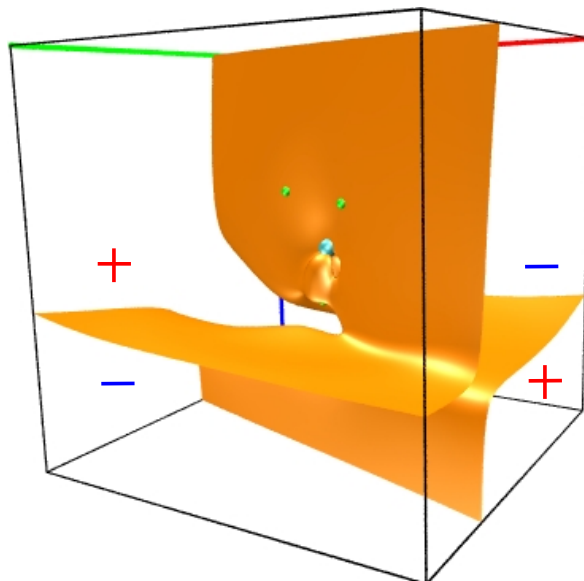
Pfaffians describe nodes more efficiently

**Nodes of different wfs (%E_corr in DMC):
atom wf scanned by 2e- singlet (3D node subset) →
correlation leads to different topologies**

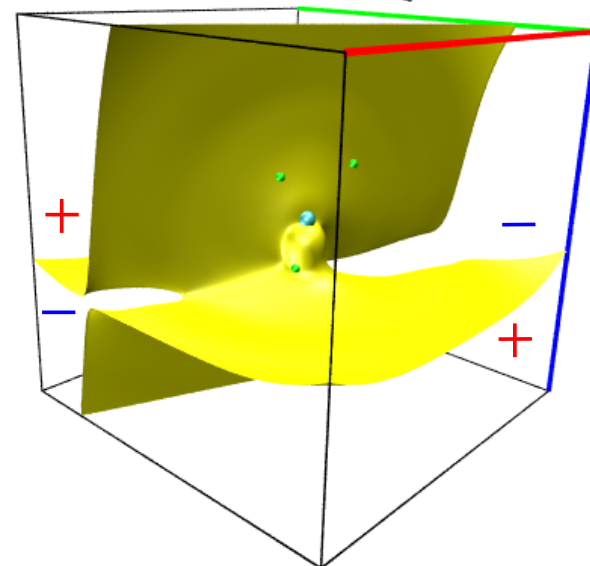
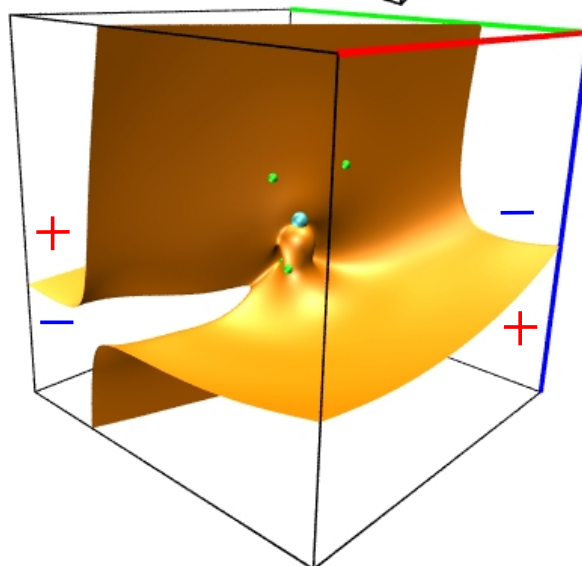
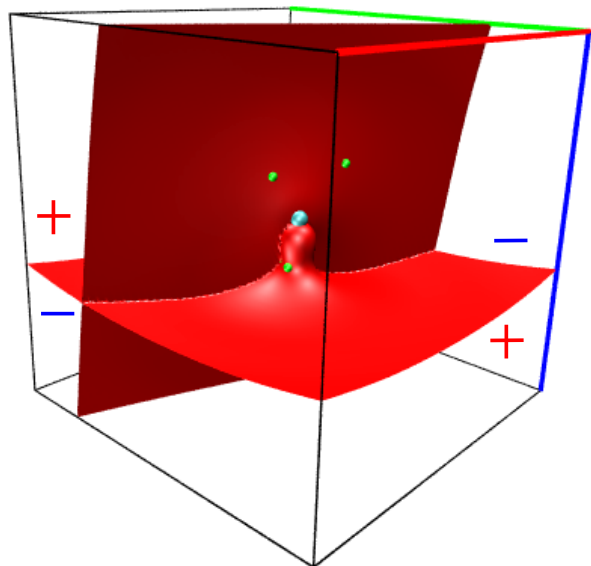
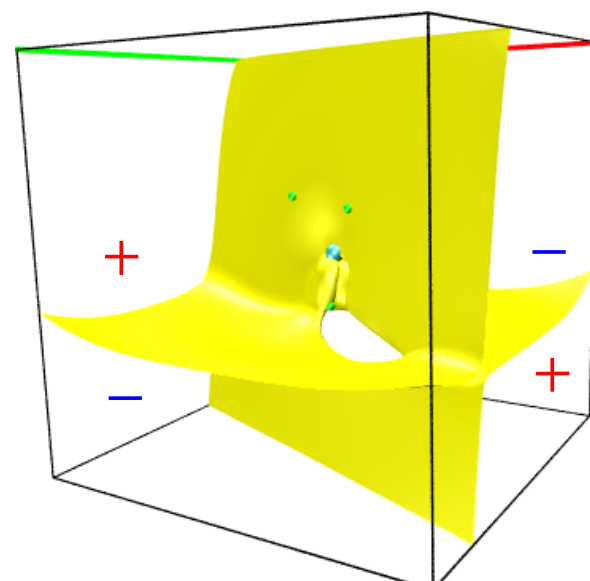
HF (94.0(2)%)



MPF (97.4(1)%)



CI (99.8(3)%)



Something different from electron-ion Hamiltonians:

Unitary ultracold atoms

Ultracold fermions system and interactions

- periodic boundary conditions

- interaction tuned to unitarity $V(r_{ij}) = -2 \frac{(2/R_{eff})^2}{\cosh^2(2r_{ij}/R_{eff})}$

- extrapolated to $R_{eff} \rightarrow 0$

- trial function $\Psi_T = \Psi_{BCS} = \det[\phi^{\uparrow\downarrow}]$

- pairing orbital built from superposition of gaussians+neighb. cells

$$\phi^{\uparrow\downarrow}(r_{ij}) = \sum_{a,b,c} \sum_k c_k^{abc} \exp[-\alpha_k^{abc} (r_{ij} - a L_x - b L_y - c L_z)^2]$$

Ultracold fermions in a special state: **unitary gas** (total energy first calculated by QMC)

Effective, short-range attractive interaction

Scattering length: a

$$1/a < 0$$

BCS, weakly paired superconductor

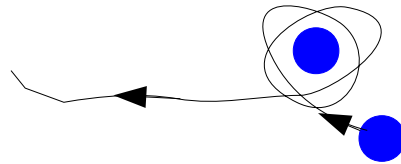
$$1/a > 0$$

BEC of covalently bonded molecules

$$1/a \rightarrow 0$$

unitary limit $\rightarrow r_{int} \ll r_s \ll a$,

Interaction is tuned, so that a pair is on the verge of forming a bound state, (ie, $E=0$)

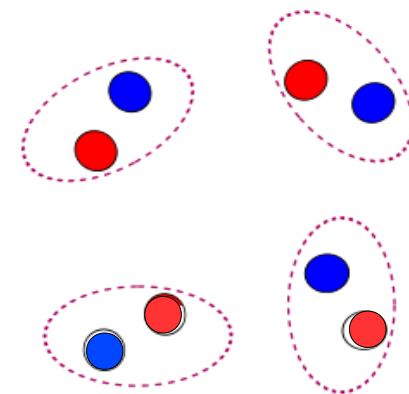
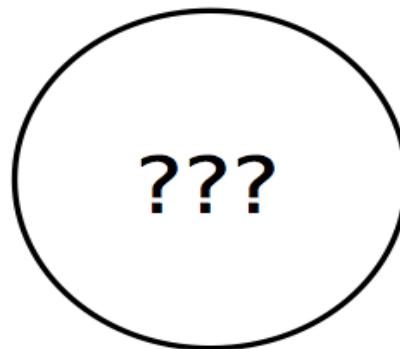
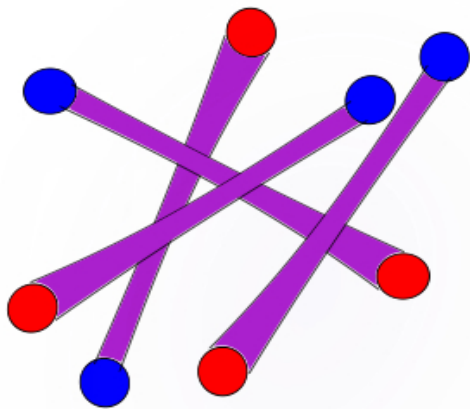
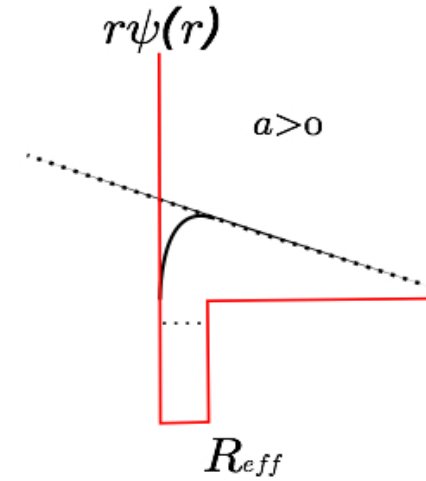
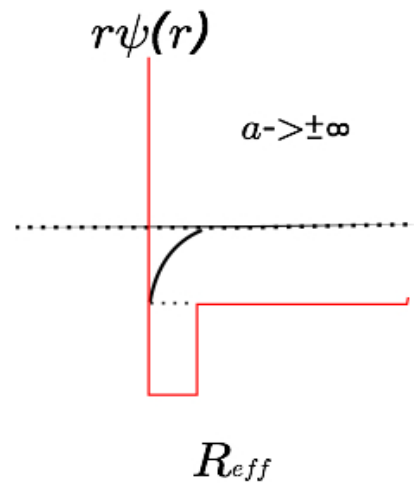
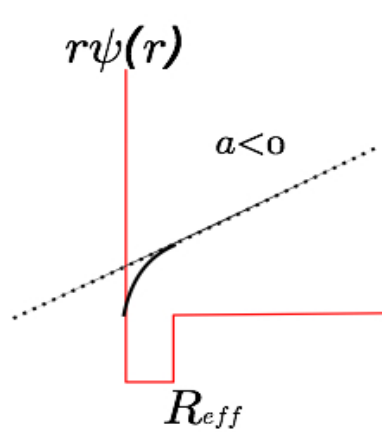


Total energy:

$$E_{tot}^{unitary} = \xi E_{tot}^{free},$$

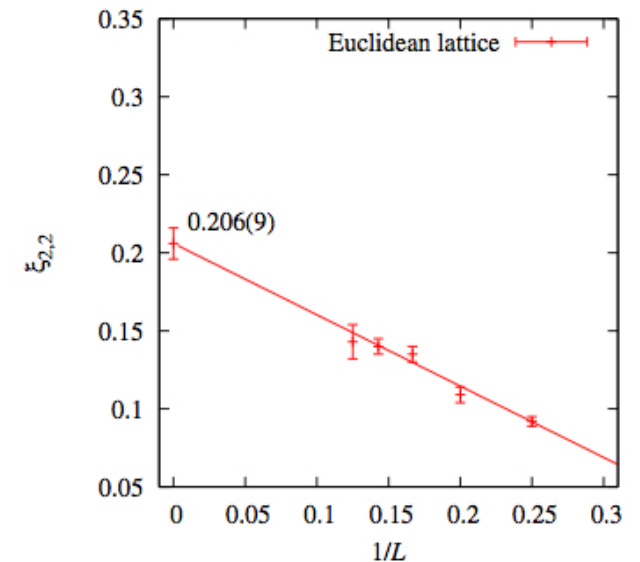
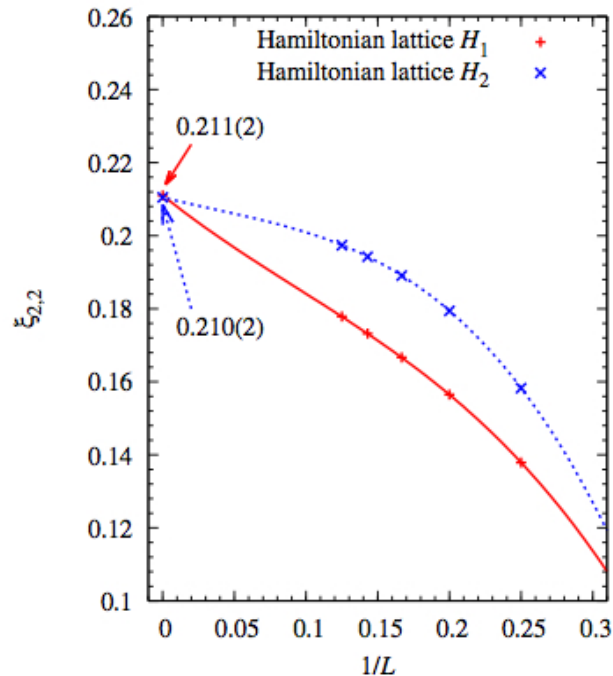
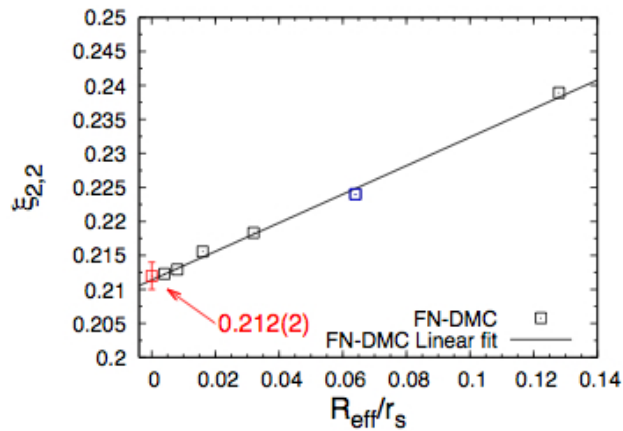
$\xi \leftarrow$ *Bertsch parameter*

BCS \rightarrow unitary \rightarrow BEC crossover



Benchmarking on a small system: 4 unitary fermions

QMC vs two independent lattice many-body methods



QMC

Direct diag.

Euclid. Lattice MC

$\xi_{2,2}$

0.212(2)

0.211(2)

0.206(9)

Bertsch parameter for (infinite) unitary fermions: improving wave functions

$$\xi_{FNDMC} / HF \text{ nodes} = 0.50(1)$$

$$\xi_{FNDMC} / BCS \text{ nodes} = 0.44(1) \quad J. Carlson et al, '03$$

$$\xi_{exact} / BCS \text{ nodes} \leq 0.393(2) \quad X. Li, L.M., PRB 2011$$

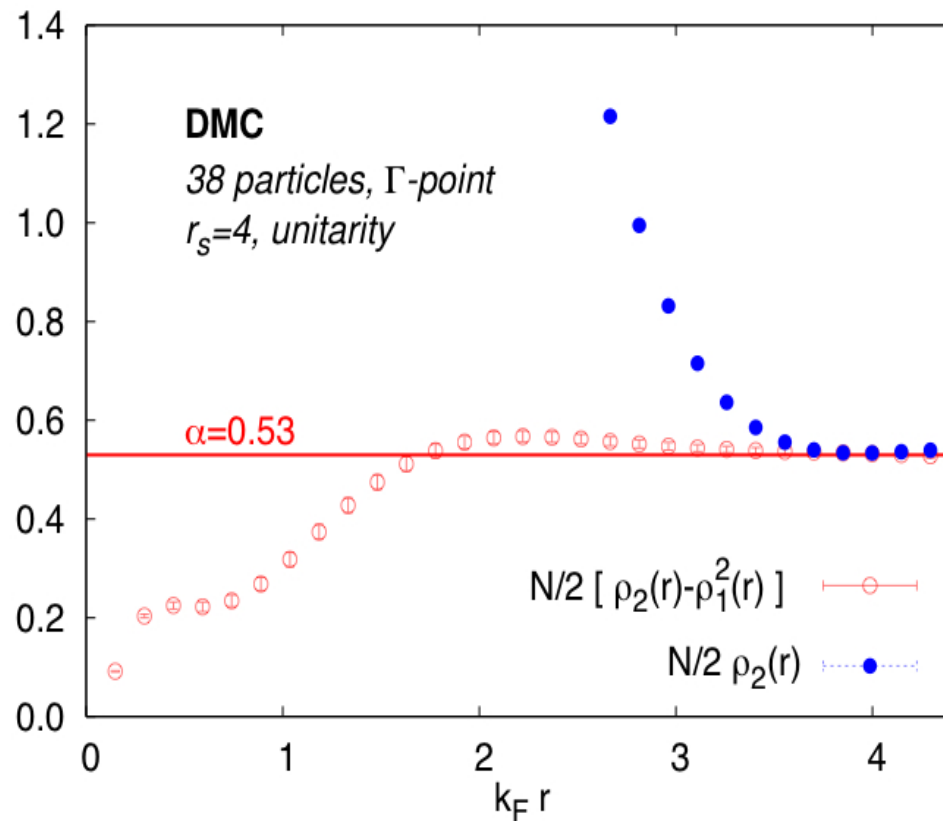
Energy gain from the BCS nodes dramatic: > 20% of the total energy !

(in electronic structure problems it is only ~ 0.03% of the total energy)

Recently, even slightly better energies were obtained, J. Carlson, S. Zhang, S. Gandolfi and coworkers 2011

Unitary limit: very robust condensate ~ 53% of particles

Find the amount of the condensate directly: averaged two-body density matrix at long-range and read off the value

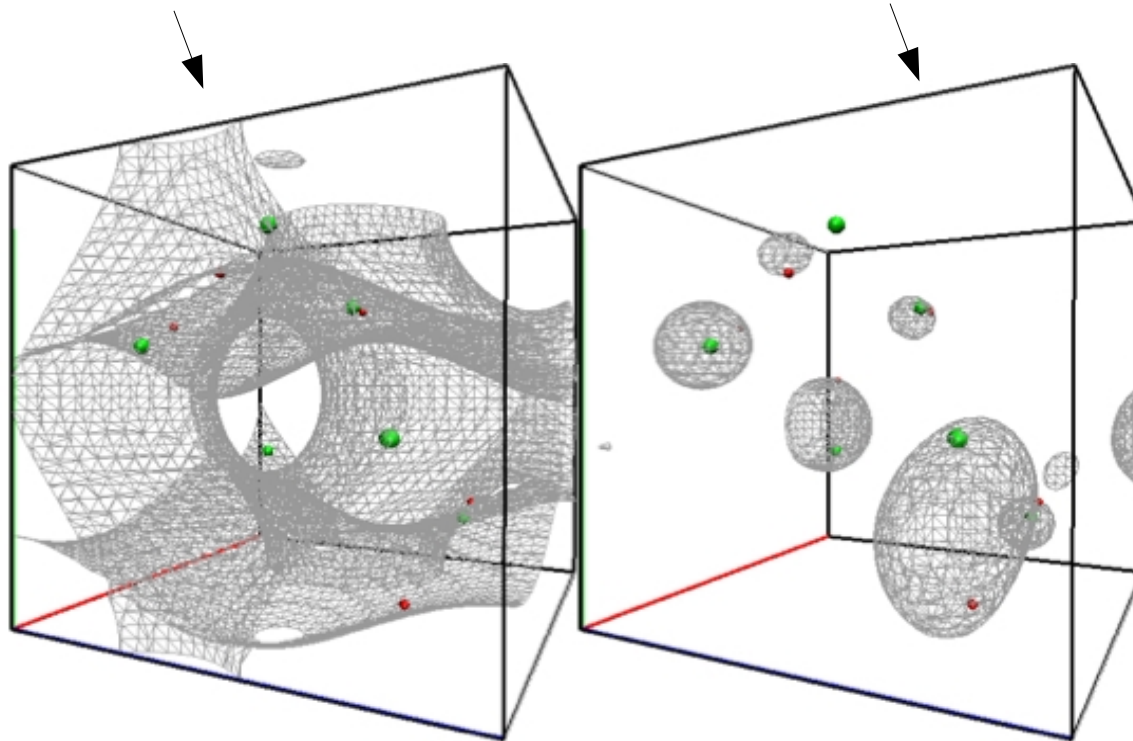


Dramatic change also in the nodes

3D scan with the spin-up and -down particle pair

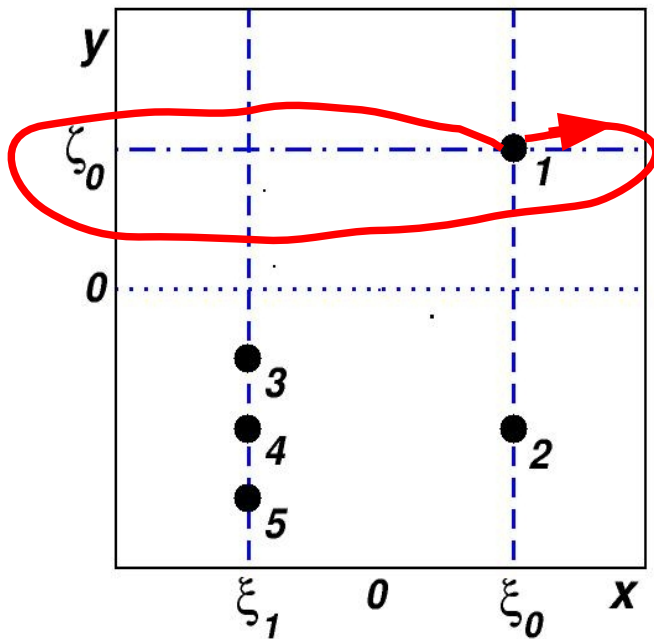
HF/free particle nodes

BCS/unitary nodes

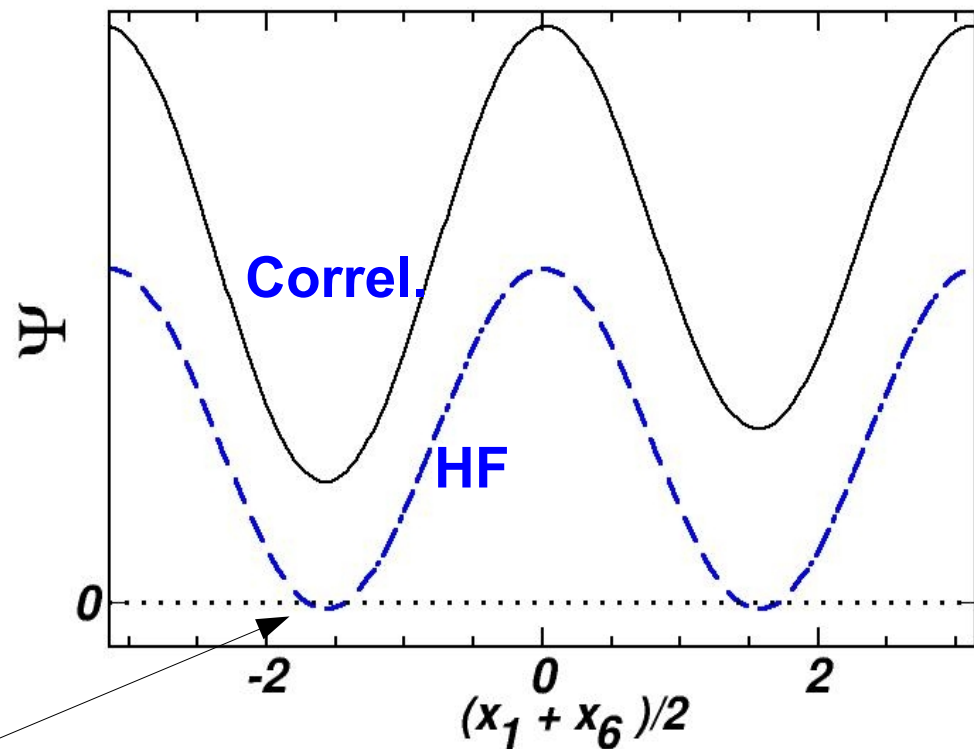


Correlated nodes in a fermion gas: singlet pair of e-winds around the box without crossing the node

$$r_i^\uparrow = r_{i+5}^\downarrow + \text{offset}, \quad i=1, \dots, 5$$



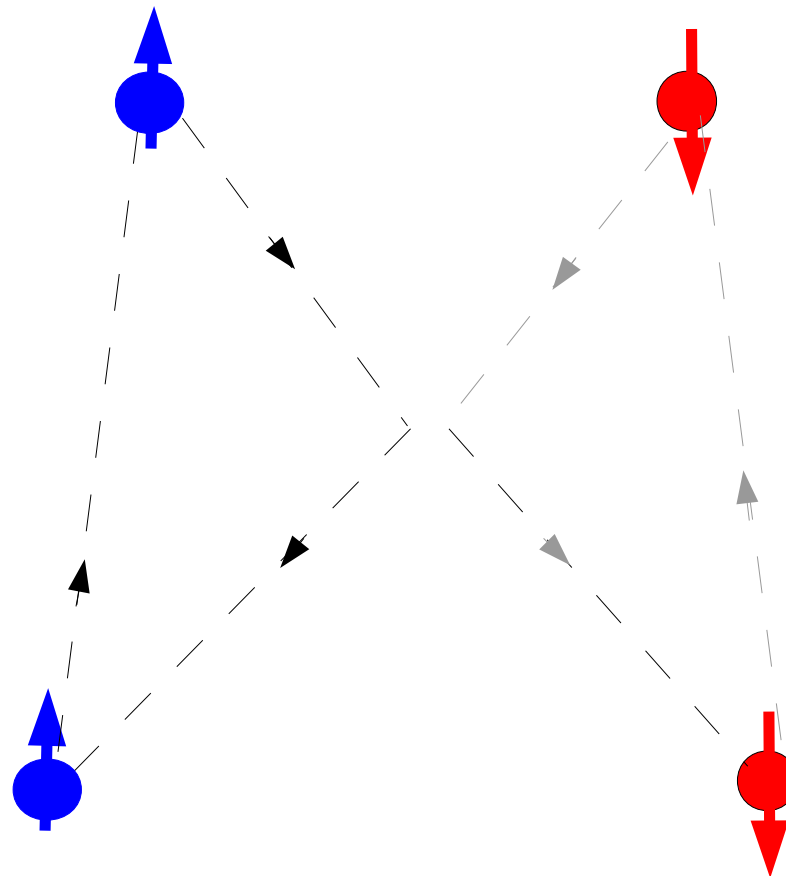
Wavefunction along the winding path



HF crosses the node, BCS/pfaffian does not (supercond.)

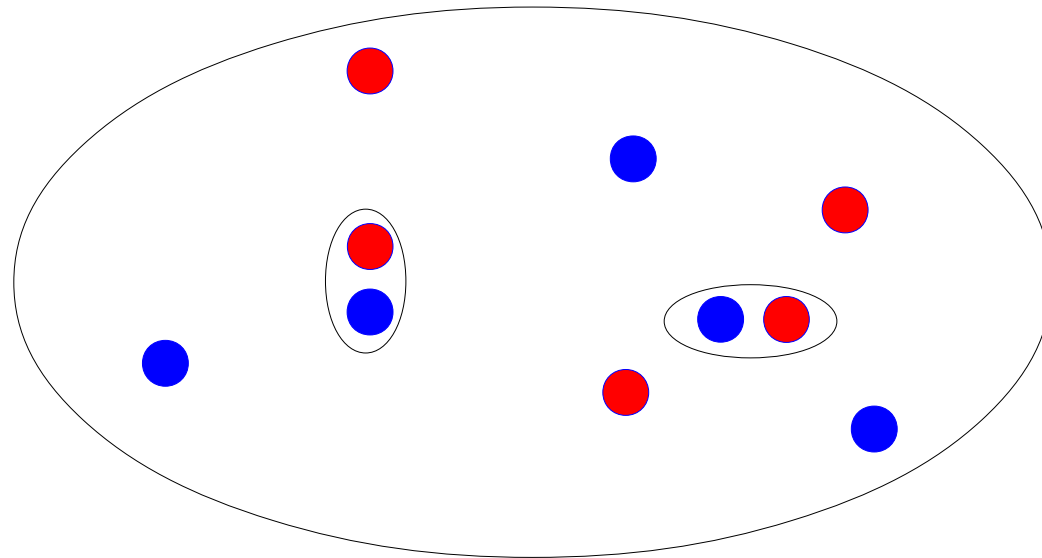
Four particle exchange: pair two-particle **simultaneous** exchanges without node crossing

Exchange in each spin channel separately has to cross the node, concerted both spin channels exchange can avoid the node



(X. Li, LM,'09)

Ultracold unitary fermions: half of the particles in tight but “dynamical” pairs, another half essentially free



A few summaries and comments

A few key points about QMC

Practical:

- systems with hundreds of electrons are feasible
- agreement with experiment within few % (problem cornered)
- sometimes “plain vanilla” single-determinant nodes very accurate

Fundamental:

- note: no ad hoc parameters, no Hubbard U or Stoner J, etc: applicable to solids, nanosystems, BEC-BCS condensates ...
- 90-95 % of correlation is “bosonic”-like (within nodal domains), efficiently captured by algebraically scaling methods
- fixed-node approx. is the only key issue: 5-10% of correlation → enough accuracy for cohesion, gaps, optical excitations, etc
- 5-10% still important: magnetic effects, superconductivity, etc

QMC and its role in electronic structure of materials (and computational science)

- +) history of important **benchmarks** (eg, correlation energy of homog. el. gas); **universally applicable** → ultracold atomic condensates with short range attractive interaction, models, etc
- /+) so far computationally **100-1000 times slower** than mainstream → helped by easier parallelization and favorable scaling in # of e-
-) **hidden cost on human time** (need to have not only the code but further **infrastructure** to make construction of many-body physics efficient → **almost nobody works in such detail**)
- +++) **QMC produces not only numbers but also understanding** (**unitary fermions insights, some more on friday talk**)

Challenges in QMC

Spectra:

- state by state calculations (band edges, band structure scan)
- subtle features (satellites, etc, difficult, costly)
- multi-state calculations (eg, so far exponential scaling)

Weakly bonded systems: difficult, energy resolution is a problem

Ionic forces: - noise is a problem → possible but costly options:

- finite differences with correlated sampling
- Hellman-Feynman, all options costly
(but AIMD/DFT with QMC correction with evolving wave function along the ionic path is surprisingly efficient, factor of 2-3 on the top of AIMD! Grossman & Mitas PRL '05)

Spins: beyond collinear spin states, spin-orbit interactions
(symposium talk)

Open Source QMC code QWalk (“Quantum Walk”)

www.qwalk.org

- **molecules and solids** (3D periodicity), 1D rings, other systems (effective interactions, model systems, etc), tested on TMOs
- **variety of basis** (gaussian, Slater, PW, numerical, etc) or combination
- **several types of correlated wavefunctions** (CI, pfaffians)
- **variety of methods** (variational, fixed-node DMC, reptation, upper bound for nonlocal operators, optimizations, etc)
- **object-oriented code, C++, 50,000+ lines: GPL open source, community** (L. Wagner, M. Bajdich, J. Kolorenc, L.M. others)
- **interfaces and converters from GAMESS, CRYSTAL, Gaussian, SIESTA, (Qespresso in progress)**

Review papers:

RMP, 73, 33 ('01)

Acta Physica Slovaca 59, 81 ('09)

Rep. Prog. Phys. 74, 026502 ('11)