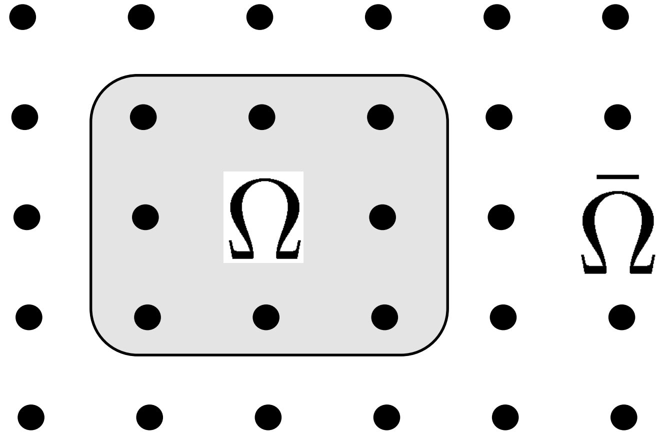
Entanglement Entropy in Conventional and Topological Orders

Shunsuke Furukawa
Condensed Matter Theory Lab., RIKEN

in collaboration with

Gregoire Misguich
Vincent Pasquier
Service de Physique Theorique, CEA Saclay, France

Introduction: What is entanglement entropy?



$|\Psi\rangle$: ground state of the total system

Reduced density matrix

$$\rho_\Omega = \mathrm{Tr}_{\bar{\Omega}}\ |\Psi\rangle\langle\Psi|$$

Entanglement entropy (von Neumann entropy)

$$S_\Omega = -\mathrm{Tr}\ \rho_\Omega \ln \rho_\Omega$$

Measure of entanglement between two regions

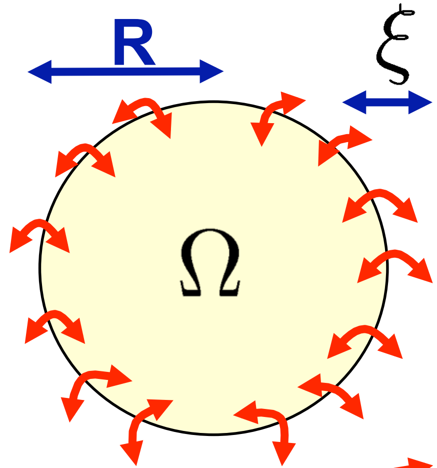
(In particular, $S_\Omega = 0$ for $|\Psi\rangle = |\psi_1\rangle_\Omega |\psi_2\rangle_{\bar{\Omega}}$)

Look at the scaling of S_Ω

→ relevant info on the correlations

Vidal, Latorre, Rico, & Kitaev, PRL, 2003

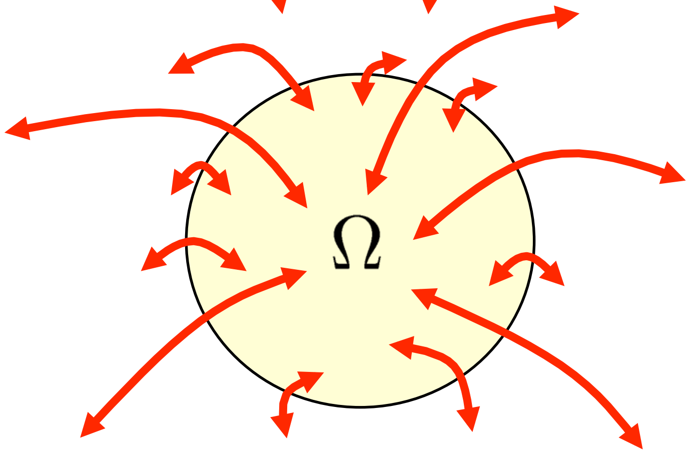
Scaling of entanglement entropy



- Short-range correlations only

$$S_\Omega \approx \alpha R^{d-1}$$ *boundary law*

Srednicki, PRL, 1993
Wolf,Verstraete,Hastings,Cirac,PRL,2008



- Power-law decaying correlations

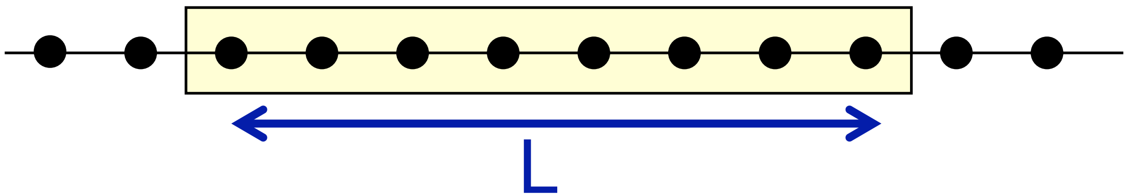
free fermion: $S_\Omega \approx \alpha R^{d-1} \ln R$

Wolf, PRL, 2006; Gioev & Klich, PRL,2006

1D critical system:

$$S_L \approx \frac{c}{3} \ln L + s_0$$

c: central charge

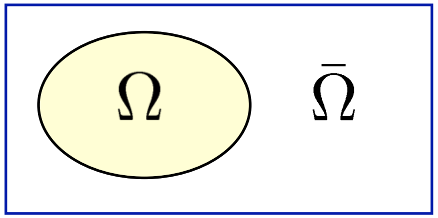


Vidal, Latorre, Rico, and Kitaev, PRL, 2003

Also in gapped systems, ⋯

$$S_\Omega = \text{const.} \times (\text{boudary size}) + (\text{universal constant})$$

Basically, boundary law



useful fingerprint of non-trivial correlations

Plan of my talk

- Topological order in 2D

  Negative constant $-\ln D_{\text{topo}}$ Kitaev & Preskill; Levin & Wen, PRL,2006

  Numerical demonstration in a quantum dimer model

- Conventional order associated with symmetry breaking

  Positive constant $\ln D_{\text{deg}}$ (D_{deg} :GS degeneracy)

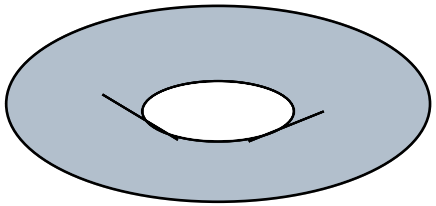
Entanglement Entropy and Z_2 Topological Order in a Quantum Dimer Model

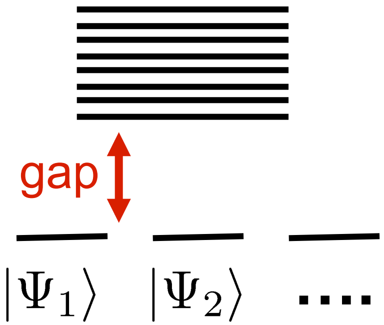
$$S_\Omega = \alpha L - \ln D_{\mathrm{topo}}$$

SF, G. Misguich, Phys. Rev. B 75, 214407 (2007)

What is topological order ?

Simply speaking, an order beyond Landau-Ginzburg paradigm





Degenerate ground states below a gap

On a torus,

degeneracy=3 for nu=1/3 FQH state

=4 for Z_2 spin liquid

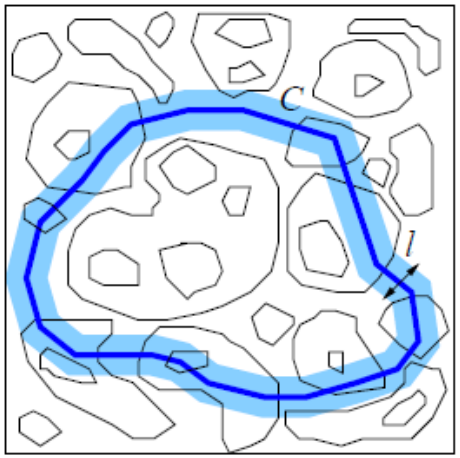
(e.g., Kitaev model, quantum dimer model)

No local order parameter can distinguish them.

$$|\langle\Psi_1|\mathcal{O}|\Psi_1\rangle - \langle\Psi_2|\mathcal{O}|\Psi_2\rangle| \sim e^{-N/\xi}$$

N: linear system size

SF, Misguich, Oshikawa, PRL,2006;J.Phys.C, 2007

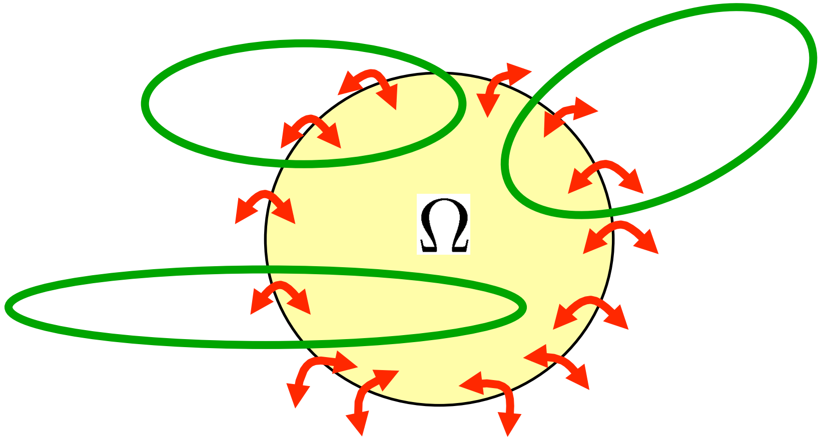


String correlations

$$\langle W(C)\rangle = \langle \prod_{i\in C} \sigma_i^x \rangle = 1$$

C: closed loop Hastings & Wen, PRB, 2005

Entanglement entropy in topological order



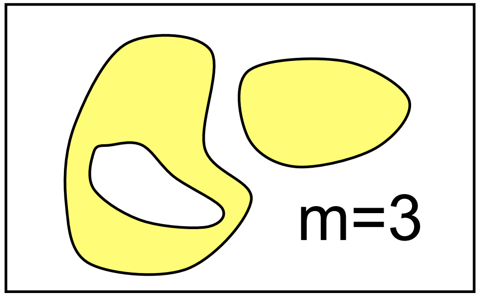
$$S_\Omega = \alpha L - \gamma$$

L: perimeter

Kitaev & Preskill, PRL 96, 110404 (2006)
Levin & Wen, PRL 96, 110405 (2006)
(Also, Hamma et al., PRA, 2005)

$$S_\Omega = \alpha L - m\gamma$$ for general cases

m: number of disconnected boundaries



$\gamma = \ln D_{\rm topo}$ **: topological entanglement entropy**

universal constant characterizing topological order

$D_{\rm topo}$ **: total quantum dimension**

If the system is described by a *discrete* gauge theory (e.g. Z_n),
D_{topo} = (number of elements of the gauge group).

Our study: Numerical demonstration of the proposal

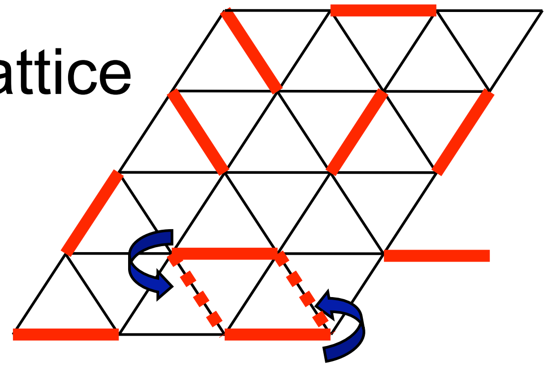
- Why is numerical check necessary?

In the original papers, an idealized situation was considered.

solvable models, correlation length = 0, no finite-size effect

Numerical check is a step toward more general cases.

- Quantum dimer model on the triangular lattice



$$H = \sum_{\square} [-t(|=\rangle\langle //| + h.c.) + v(|=\rangle\langle =| + |//\rangle\langle //|)]$$

Moessner & Sondhi, PRL,2001

Rokhsar-Kivelson point (t=v)

$$|\mathrm{RK}\rangle = \frac{1}{\sqrt{|\mathcal{E}|}} \sum_{c \in \mathcal{E}} |c\rangle$$

equal-amplitude superposition of all dimer configs.

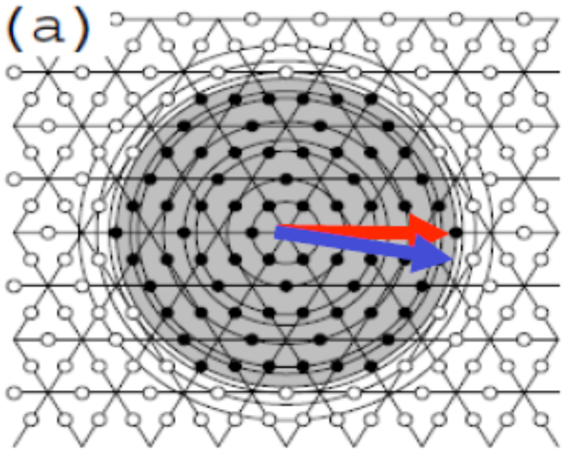
- example of Z2 topological order

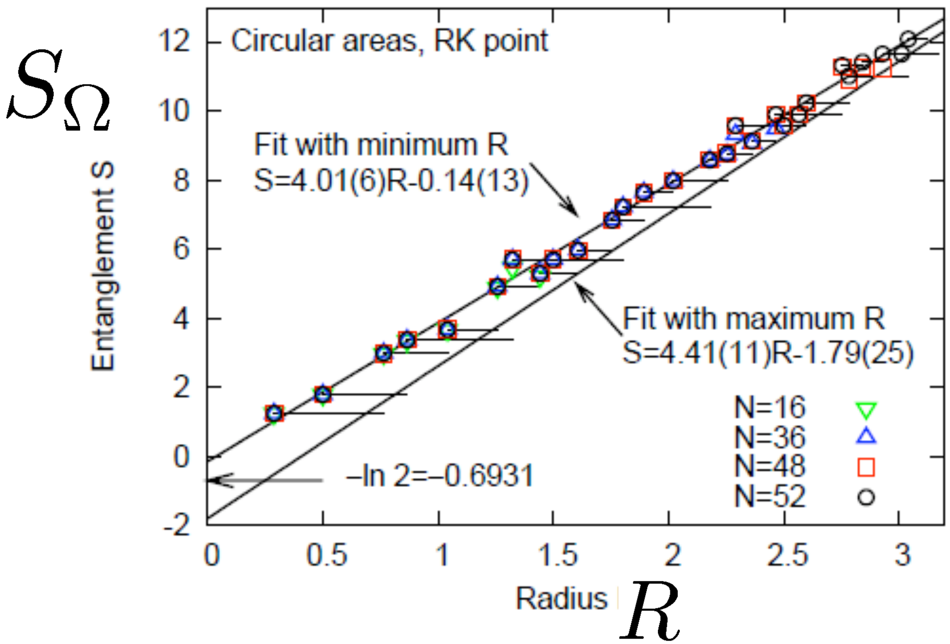
⇒ $\gamma = \ln 2$ is expected

- finite correlation length

⇒ Finite-size effects arise.

Entanglement entropy on circular areas





Linear relation $S = \alpha R + \text{const.}$ is observed.

But O(1) ambituity on R ($R_{min} \neq R_{max}$)

A special procedure for extracting topological term.

Extraction of topological term

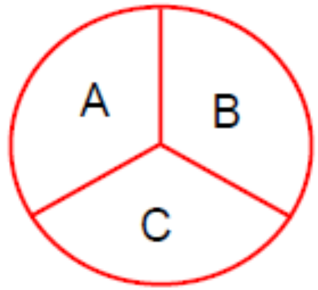
$$S_\Omega = \alpha L - \gamma$$

Take linear combination of entanglement entropies

⇒ Cancel out all the boundary terms.

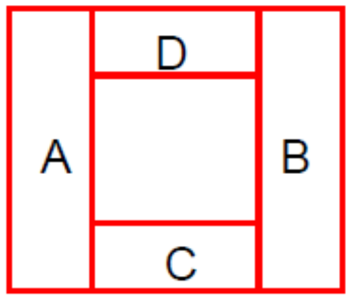
◆ Kitaev-Preskill construction

$$S_{ABC} - (S_{AB} + S_{BC} + S_{AC}) + (S_A + S_B + S_C) \longrightarrow -\gamma$$



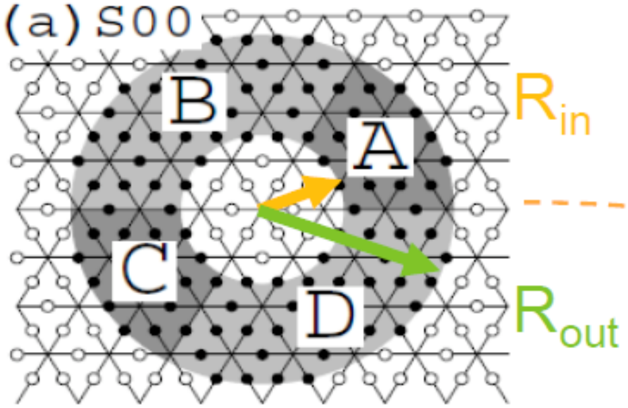
◆ Levin-Wen construction

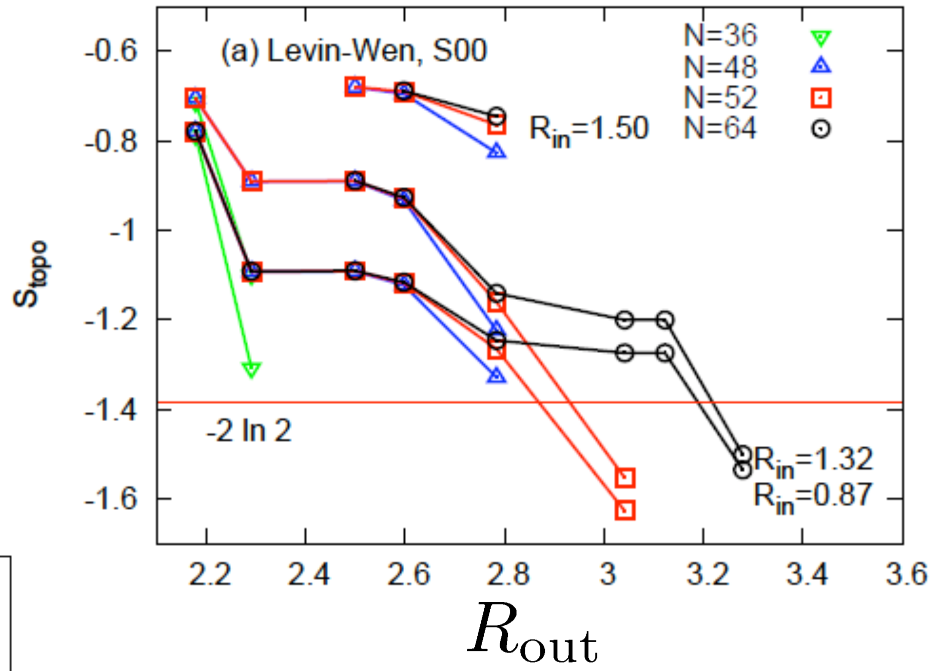
$$S_{ABCD} - S_{ABD} - S_{ABC} + S_{AB} \longrightarrow -2\gamma$$

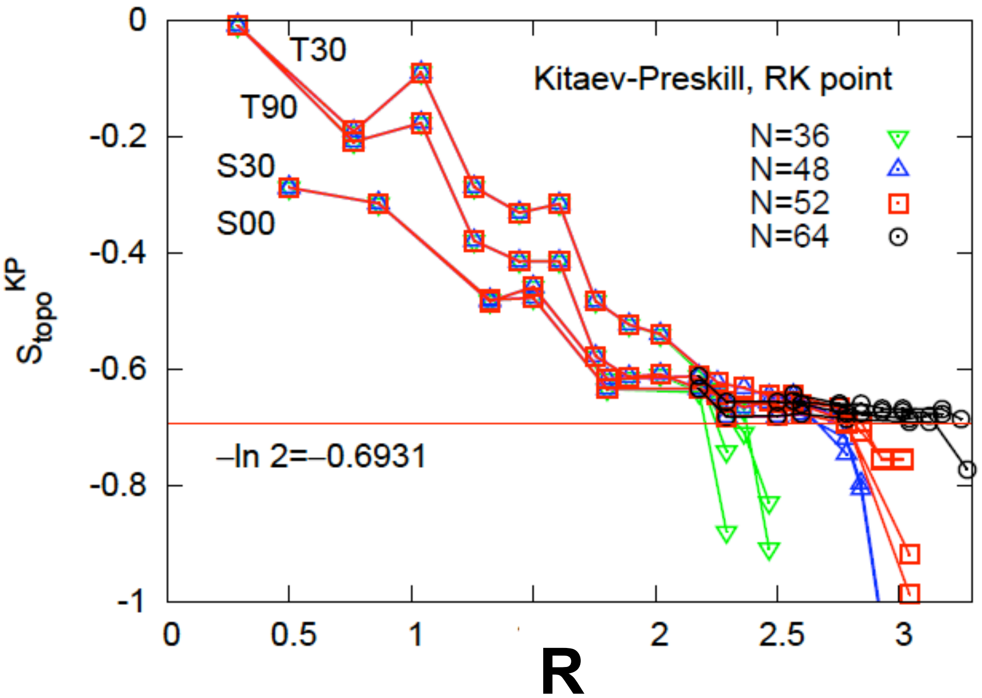


Topological entanglement entropy – numerical result

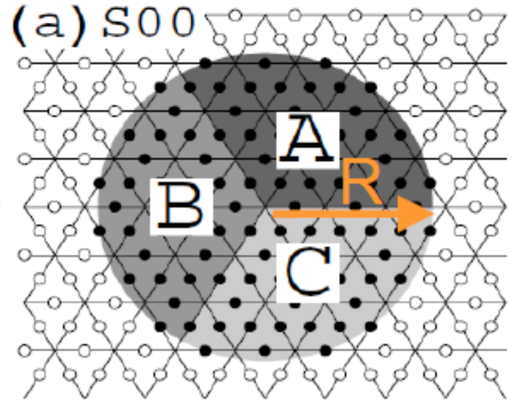
Levin-Wen $\longrightarrow -2\gamma$





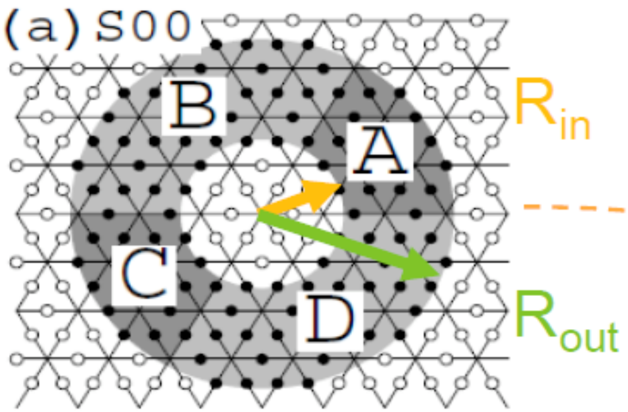


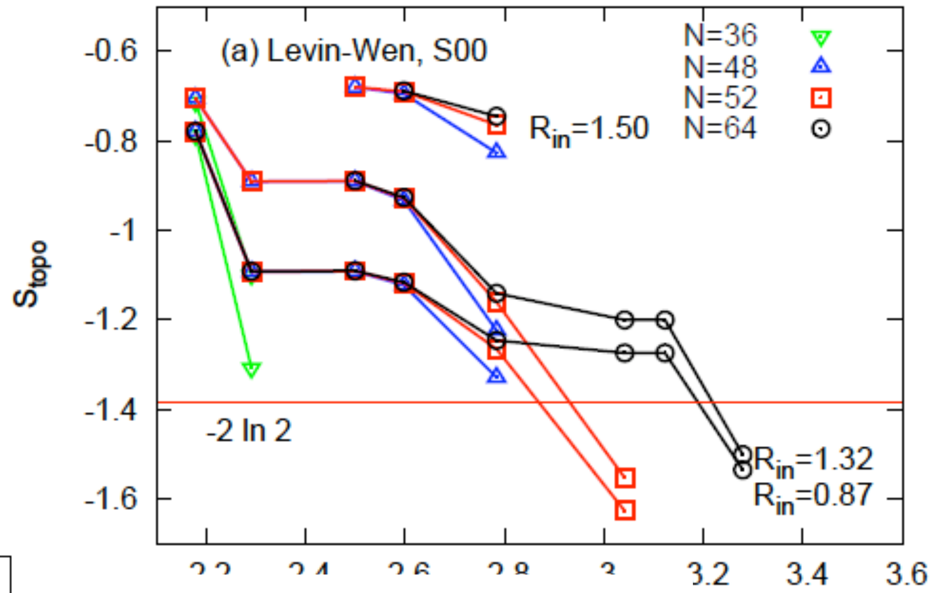
Kitaev-Preskill $\longrightarrow -\gamma$

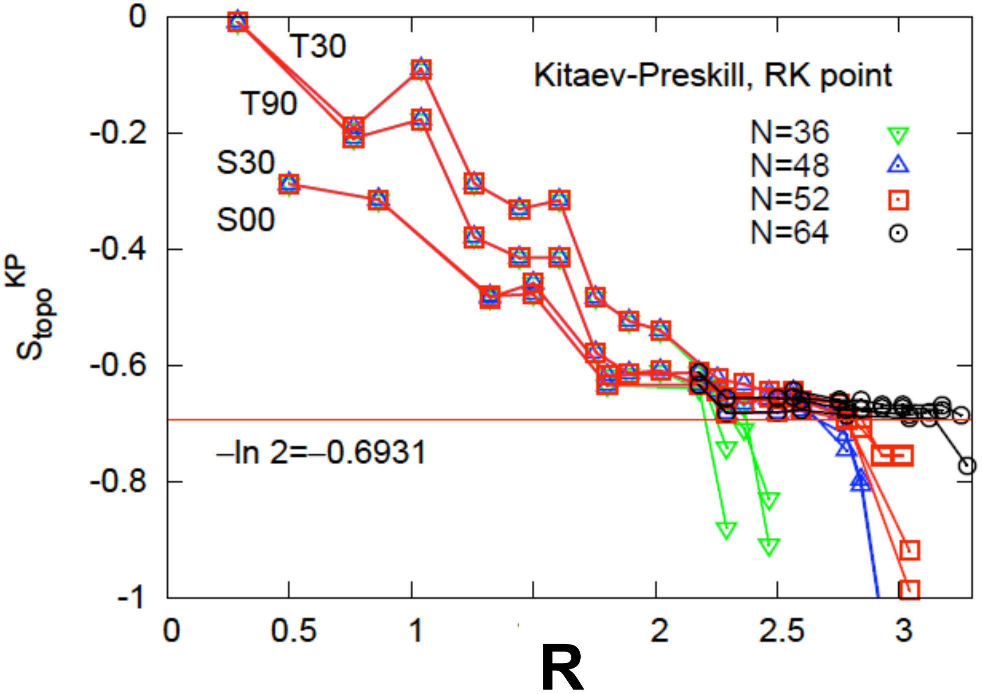


Topological entanglement entropy – numerical result

Levin-Wen





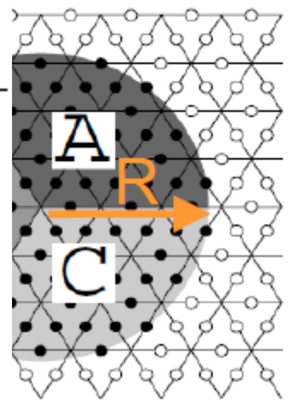


99% agreement with the prediction !

reskill

S00 case

Radius R	$-S_{topo}^{KP}/\ln 2$	
	$N=52$	$N=64$
2.18	0.9143	0.9143
2.29	0.9839	0.9835
2.50	0.9822	0.9822
2.60	0.9765	0.9760
2.78	1.0014	0.9897
3.04	1.3252	0.9967
3.12		0.9967



Macroscopic Entanglement and Symmetry breaking

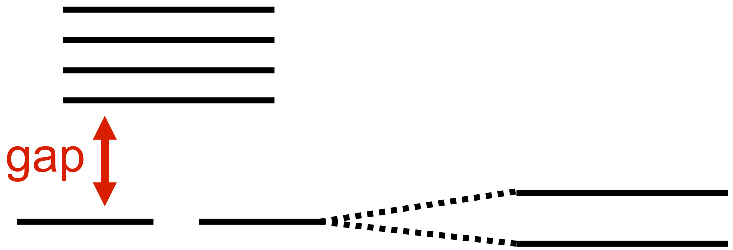
$$S_\Omega = \text{ const.} \times (\text{boudary size}) + \ln D_{\text{deg}}$$

Symmetry breaking and macroscopic entanglement

ex.) Ising model in a weak transverse field

thermodynamic limit

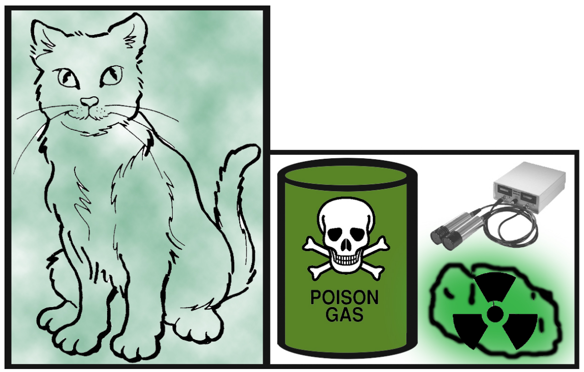
finite-size system



$|\uparrow\uparrow\uparrow\ldots\rangle \quad |\downarrow\downarrow\downarrow\ldots\rangle$

$$\frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\ldots\rangle - |\downarrow\downarrow\downarrow\ldots\rangle)$$

$$\frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\ldots\rangle + |\downarrow\downarrow\downarrow\ldots\rangle)$$



(from Wikipedia)

$$|\text{alive}\rangle + |\text{dead}\rangle$$

Superposition of macroscopically distinct states

Can we detect this structure?

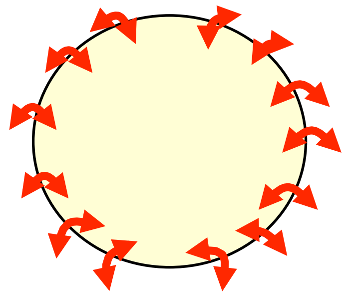
Can we count how many ``distinct'' states are superposed?

Macroscopic entanglement entropy – heuristic argument

- Pure ferromagnetic state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\ldots\rangle + |\downarrow\downarrow\downarrow\ldots\rangle)$

$$S_A = \ln 2 \quad \text{for any region A}$$

- Perturbed, e.g., by transverse field

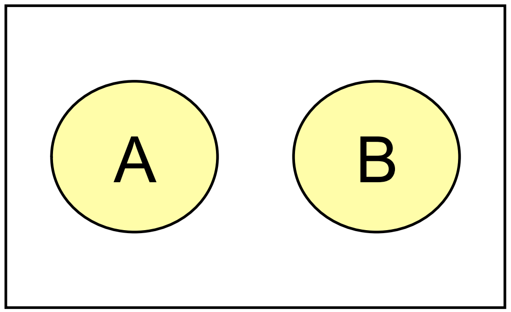


$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\ldots\rangle + |\downarrow\downarrow\downarrow\ldots\rangle) + (\text{perturbations})$$

$$\langle\sigma_0^z\sigma_r^z\rangle \approx M^2 + c\, e^{-r/\xi}$$ ← short-range correlation

$$S_A = \ln 2 + \text{const.} \times (\text{boudary size})$$ **smeared!**

cancel out



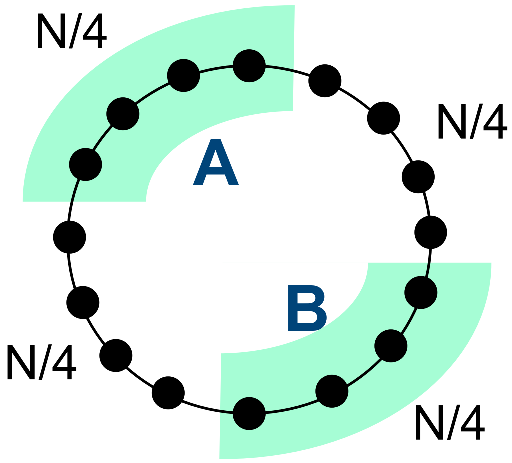
$$I_{A:B} \equiv S_A + S_B - S_{A\cup B}$$

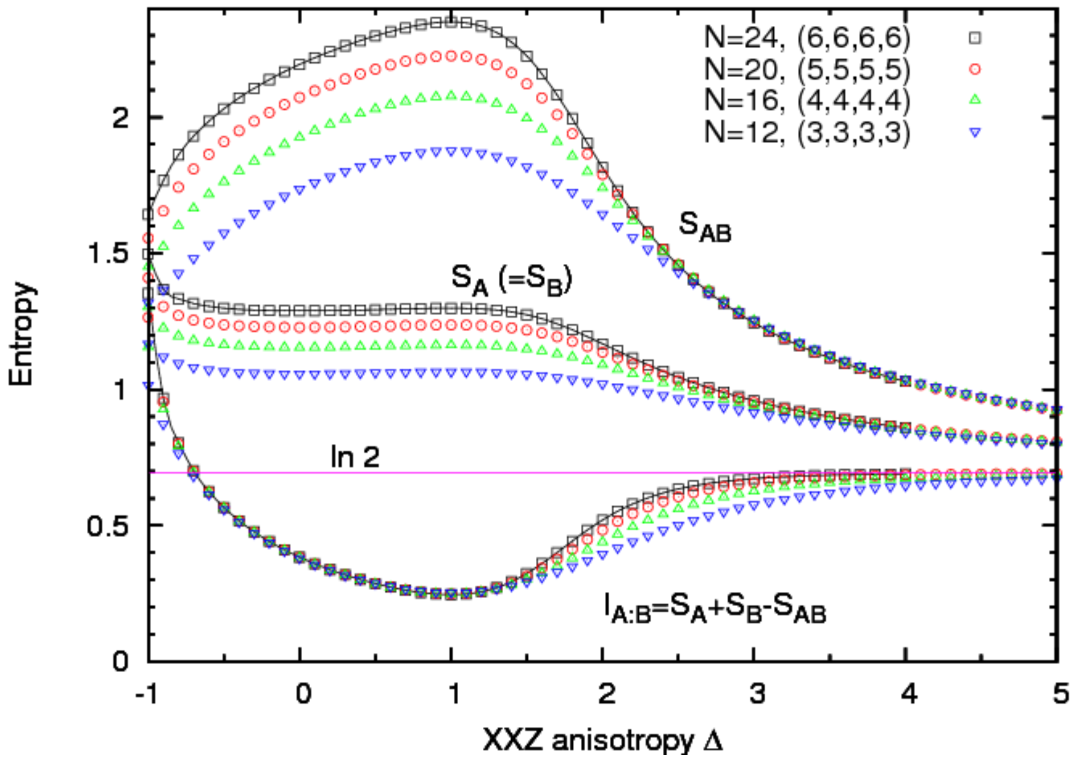
$$\longrightarrow \ln D_{\text{deg}}$$

``Macroscopic entanglement entropy''

Numerical demonstration – XXZ chain

$$H = \sum_j [S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z]$$





$$S_{A\cup B} = \ln 2 + \mathcal{B}_A + \mathcal{B}_B$$

$$\begin{cases} S_A = \ln 2 + \mathcal{B}_A \\ S_B = \ln 2 + \mathcal{B}_B \end{cases}$$

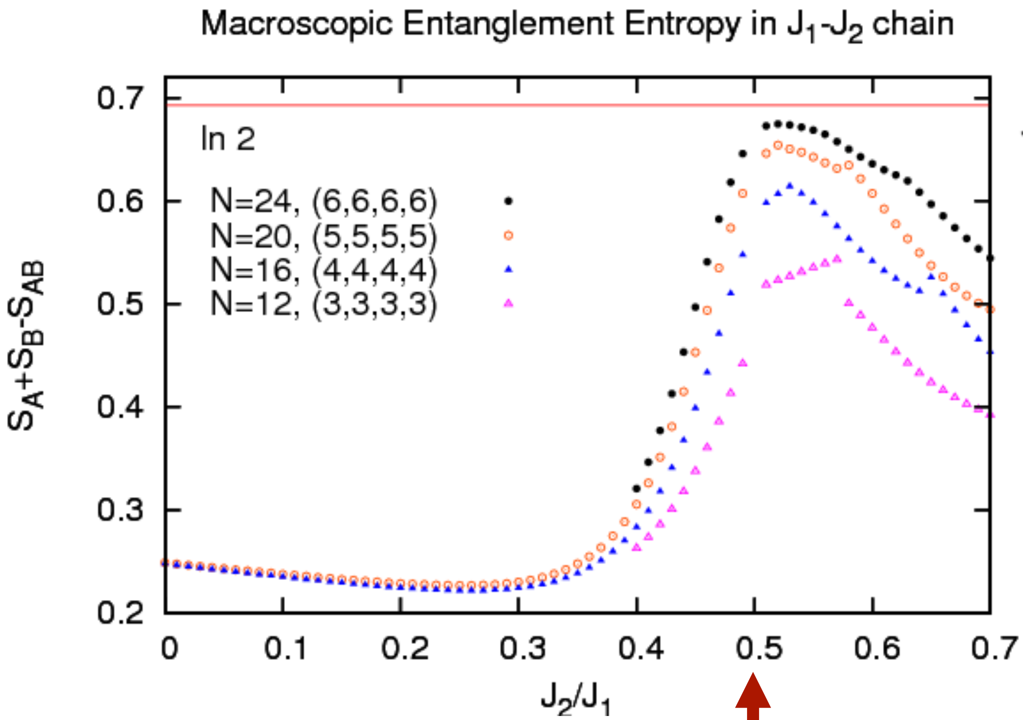
boundary terms

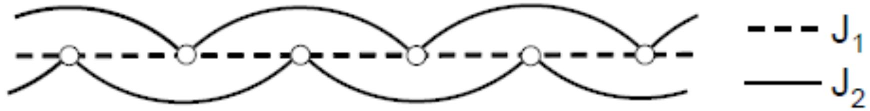
$$S_A + S_B - S_{A\cup B} = \ln 2$$

TL liquid | Neel phase

2-fold GSs

Numerical demonstration – J1–J2 frustrated chain



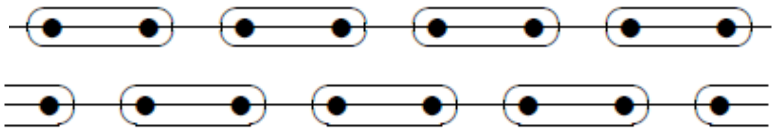


ln 2 around Majumdar-Ghosh

Applies to any sym.-broken phase irrespective of the form of order paramter.

0.241 | Majumdar-Ghosh

TL liquid | dimer phase 2-fold GSs



Unbiased probe of symmetry breaking

Combination with iTEBD (DMRG-like algorithm)

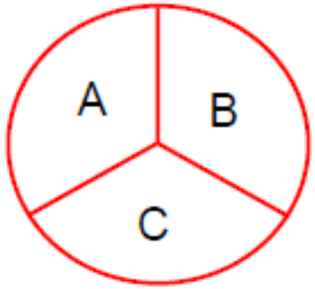
Summary

$$S_\Omega = \text{const.} \times (\text{boudary size}) + (\text{universal constant})$$

- Topological entanglement entropy in a quantum dimer model

  - One of the first numerical demonstration of Kitaev-Preskill and Levin-Wen proposal

    (other attempt: Haque et al., PRL, 2007 for Laughlin state)

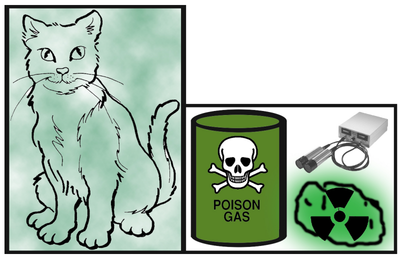
  - Negative constant $-\ln 2$

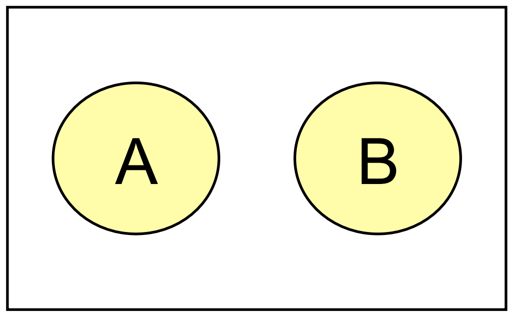
    ⇒ in agreement with Z2 topological order

- Macroscopic entanglement and symmetry breaking

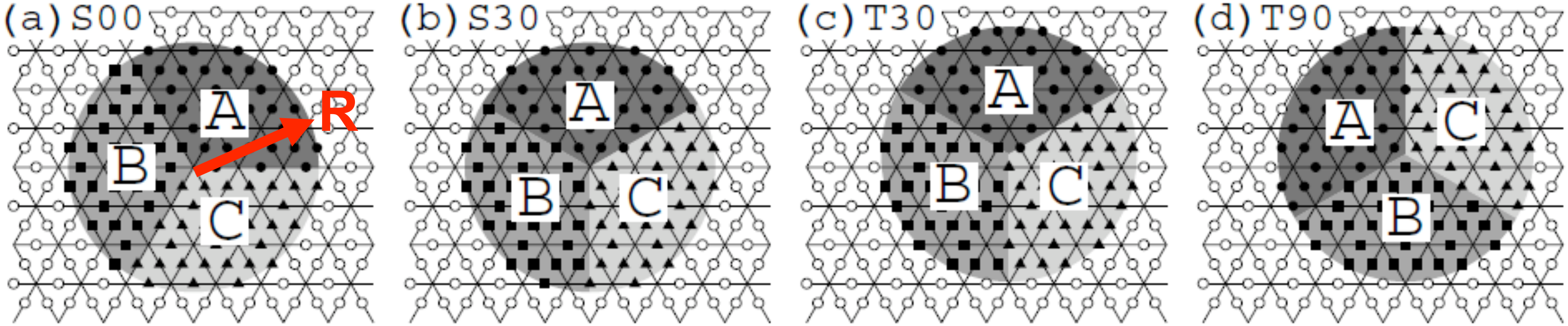
  - Positive constant $\ln D_{\text{deg}}$

  - Unbiased probe of sym. breaking





Kitaev–Preskill construction



(a), (b): site-centered, R=2.78 (c), (d): triangle-centered, R=2.83

$$S_{\text{topo}}^{\text{Kitaev-Preskill}} = S_A + S_B + S_C$$

$$- S_{AB} - S_{BC} - S_{CA} + S_{ABC}$$

S00 case		
Radius R	$-S_{\text{topo}}^{\text{KP}}/\ln 2$	
	$N=52$	$N=64$
2.18	0.9143	0.9143
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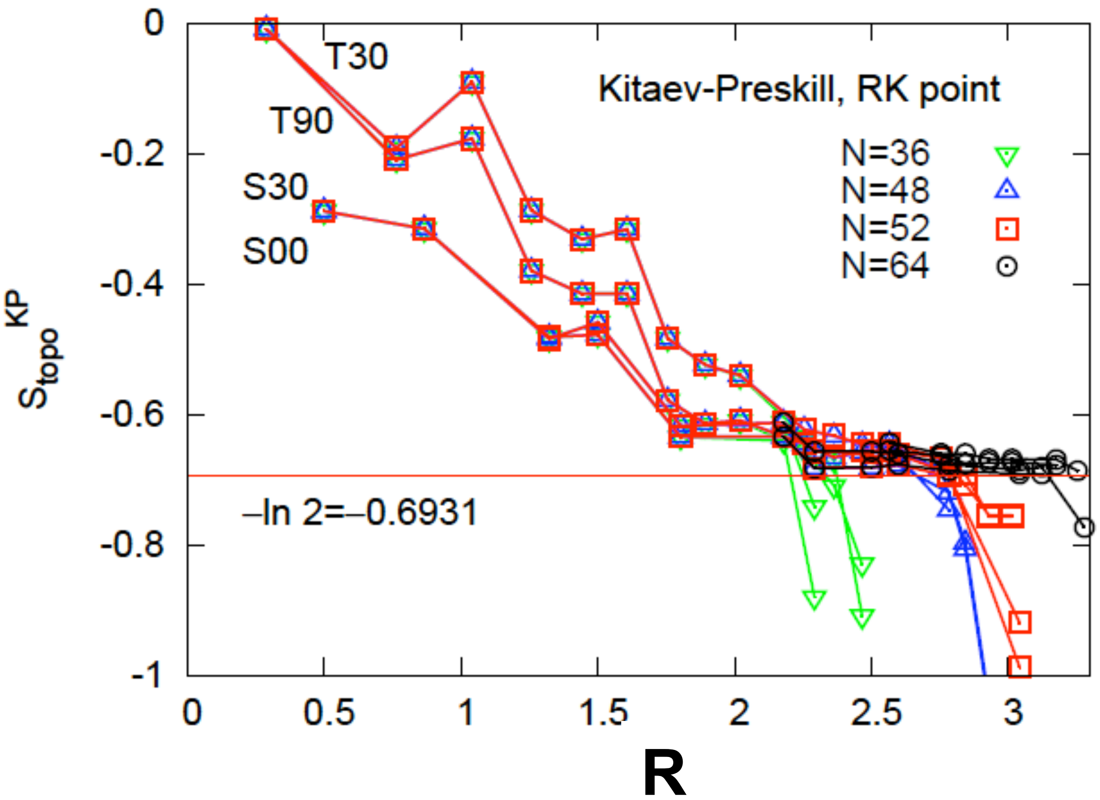
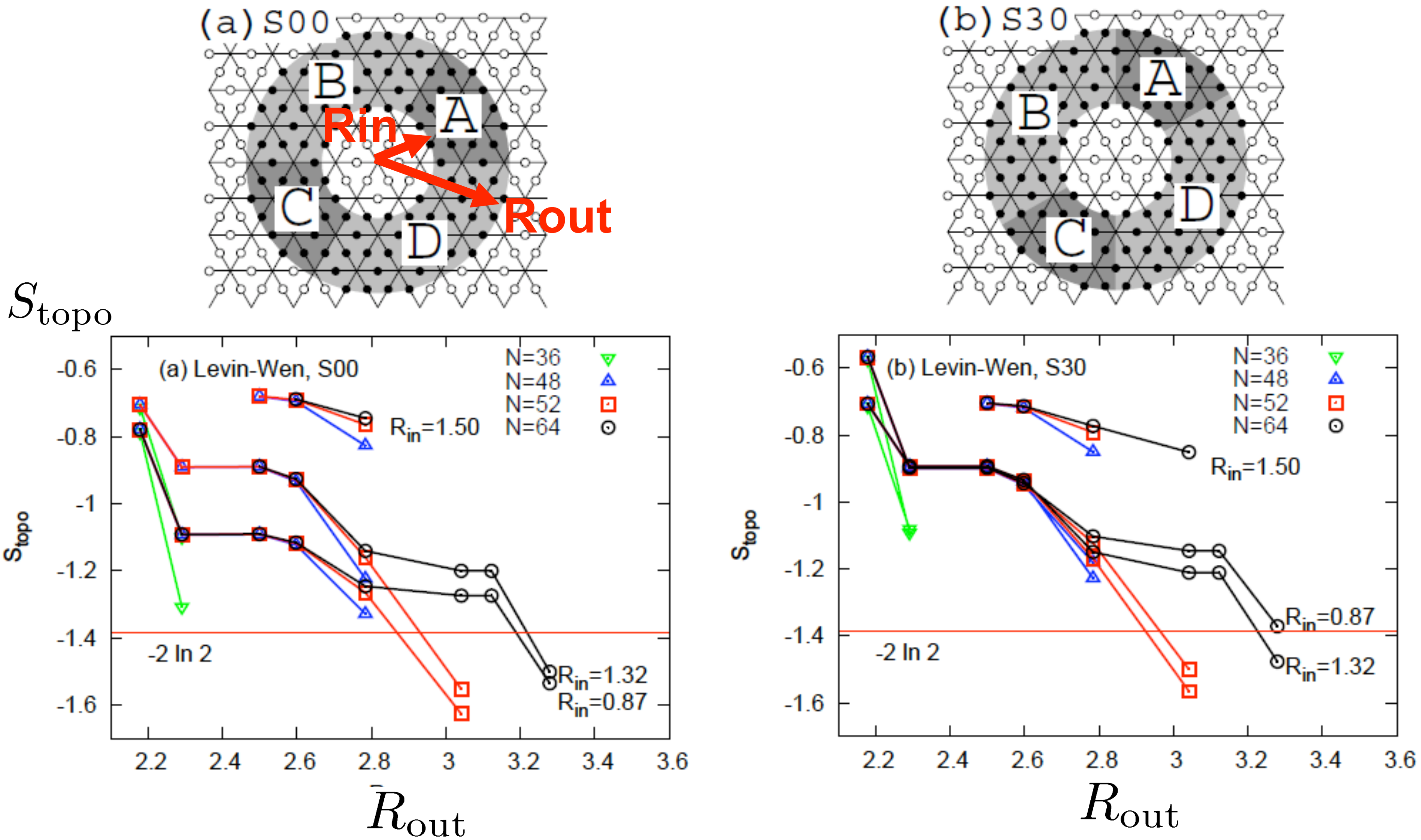
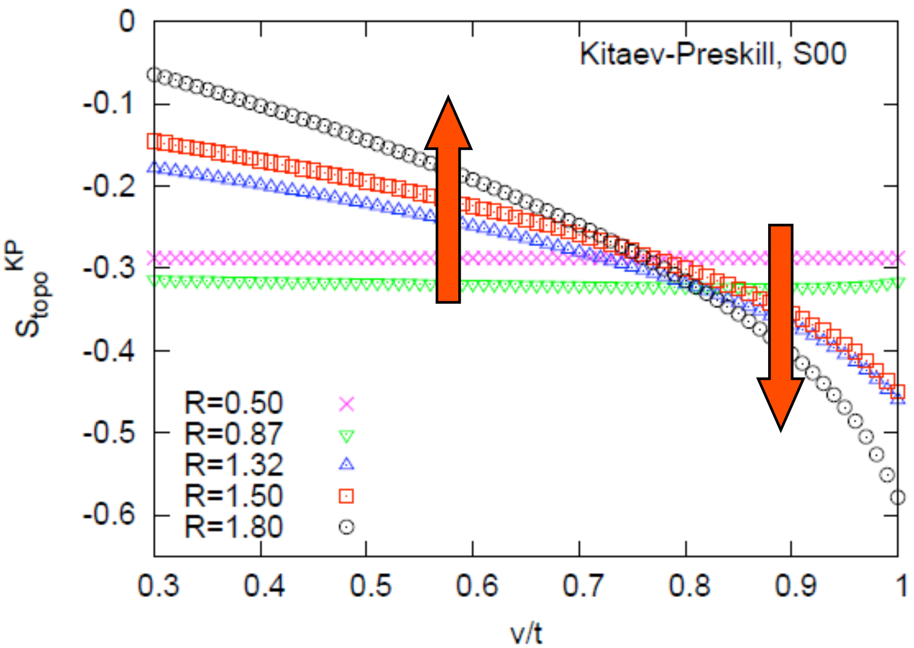


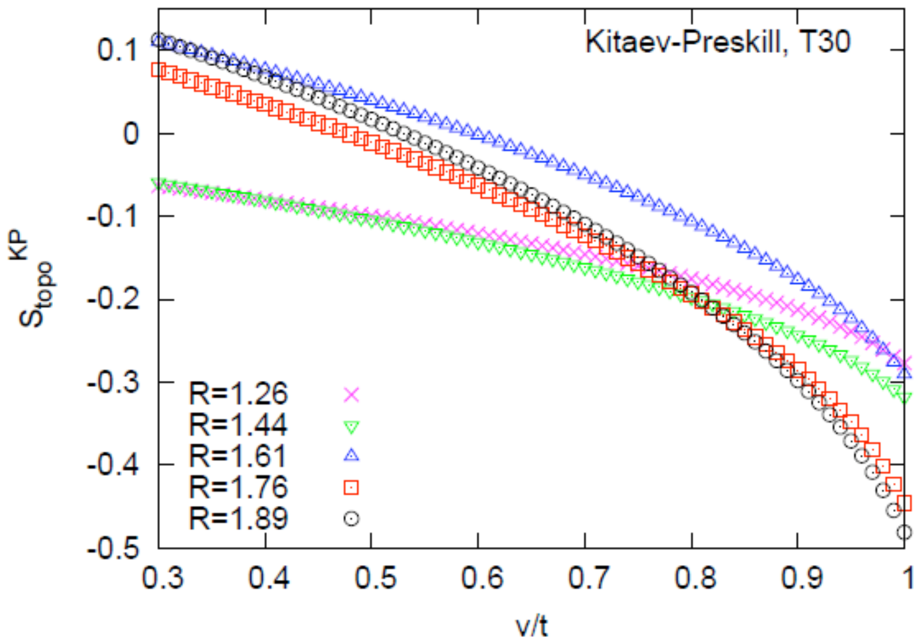
Levin–Wen construction

$$S_{\text{topo}}^{\text{Levin-Wen}} = S_{ABCD} - S_{ABC} - S_{CDA} + S_{AC} \longrightarrow -2\gamma$$



Phase transition



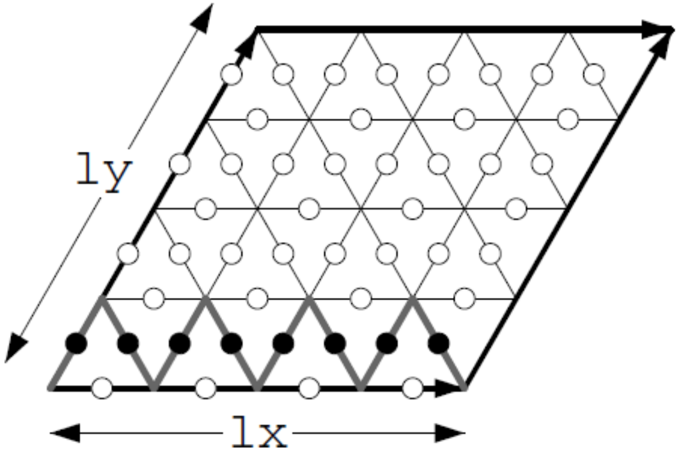


$\sqrt{12} \times \sqrt{12}$ VBC | RVB

$\sqrt{12} \times \sqrt{12}$ VBC | RVB

Tendency changes with the phase transition.

Zigzag non-local area



Motivation

Boundary length is exactly proportinal to lx.

⇩

Clear linear dependence is expected.
(using different system sizes)

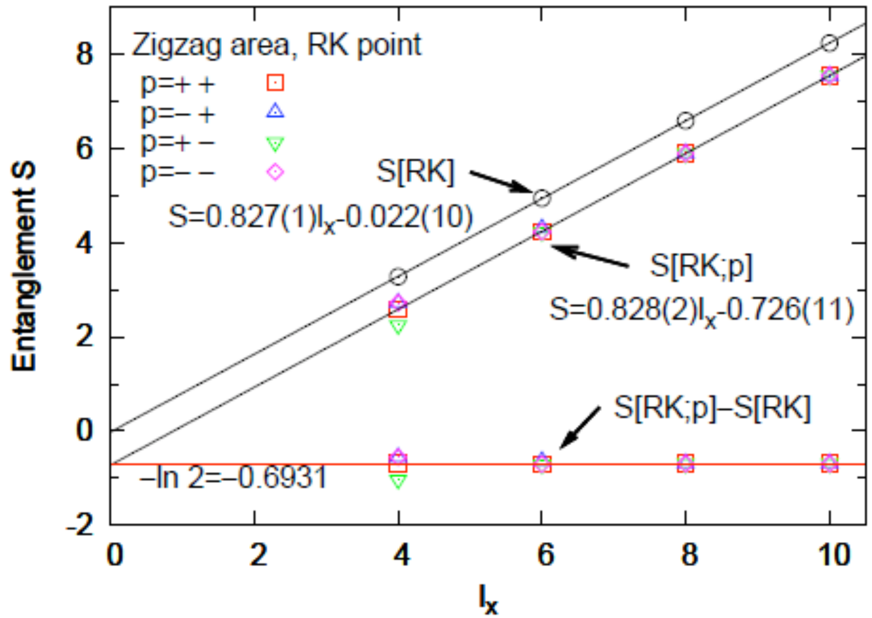
Attention: Dependence on the choice of GS

Expected result from a solvable model
(Hamma et al, PRB,05)

$$S = \alpha l_x \quad \text{for} \quad |\mathrm{RK}\rangle = \frac{1}{\sqrt{|\mathcal{E}|}} \sum_{c \in \mathcal{E}} |c\rangle$$

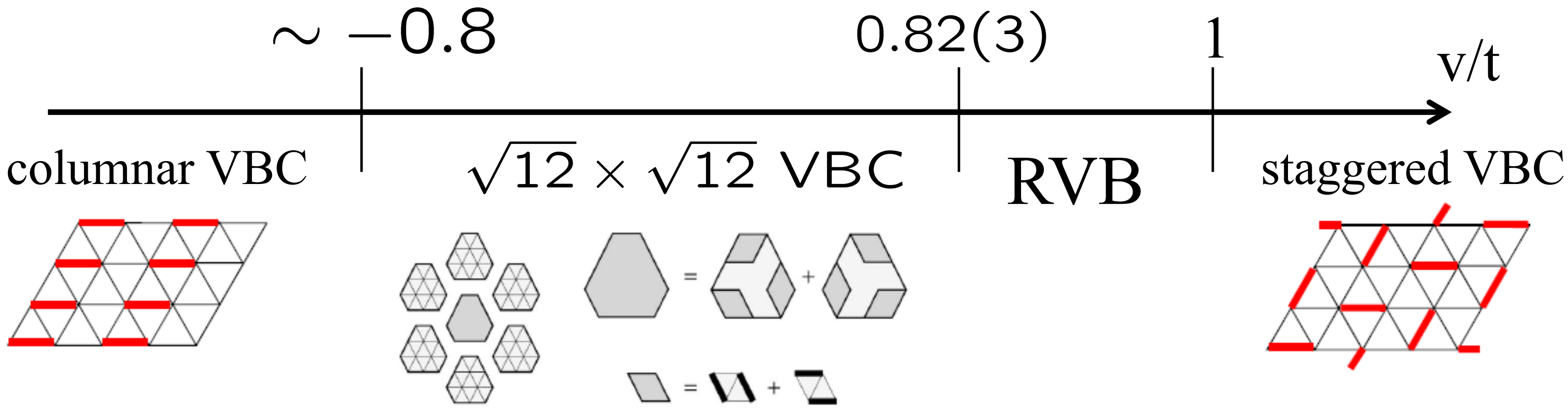
$$S = \alpha l_x - \gamma \quad \text{for} \quad |\mathrm{RK}; p\rangle = \frac{1}{\sqrt{|\mathcal{E}_p|}} \sum_{c \in \mathcal{E}_p} |c\rangle$$

	$(S[\mathrm{RK}] - S[\mathrm{RK};p])/\ln 2$			
$l_x\ (= l_y)$	$++$	$-+$	$+-$	$--$
4	1.0024*	0.8051	1.4910	0.8051
6	1.0315	0.9248	1.0315	1.0212*
8	0.9944*	1.0022	1.0017	1.0022
10	0.9981	1.0028	0.9981	1.0011*



Phase diagram

QMC: Moessner and Sondhi, PRL 86 (2001); Green fn. MC: Ralko et al., PRB 71 (2005)



Numerical methods in our analysis

- Rokhsar-Kivelson point t=v

$$|\mathrm{RK}\rangle = \frac{1}{\sqrt{|\mathcal{E}|}} \sum_{c \in \mathcal{E}} |c\rangle$$

⇒ Enumeration of dimer coverings (up to N=64)

- v/t<1 ⇒ exact diagonalization (up to N=36)