• Concept of geometric frustration and examples.
• General thermodynamic properties of frustrated magnets.
• Ground state degeneracy.
• Energy barriers and order by disorder.
• Frustration in quantum magnets.
• Quantum states: spin liquids and valence bond order.
• Dimensional reduction and fractional excitations.
• Klein models, checkerboard lattice and quantum spin-ice.
• Electrodynamics of spinons.
• Perturbations away from the optimally frustrated point.
• Summary
Geometric frustration arises when the spins cannot satisfy all the interactions simultaneously:
Examples of frustrated lattices

1D

NaCu_2O_2

Azurite: Cu_3(CO_3)_2(OH)_2

2D

CuFeO_2

SrCr_9Ga_3O_19

SrCu_2(BO_3)_2

3D

Dy_2Ti_2O_7

SrL_2O_4
Ground State Degeneracy

Classical n.n. AF Heisenberg Hamiltonian:

\[ H = J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j \]

For \( q \) mutually interacting spins:

\[ H = J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j = \frac{J}{2} L^2, \]

where \( L = \sum_{i}^q \mathbf{S}_i \).

The minimum energy condition \( L=0 \) imposes three constraints \( L_x = L_y = L_z \) for spins with \( n=3 \) components. The number of constraints is \( K=n \). The number of degrees of freedom excluding the global rotations is \( F=q(n-1)-n \). So, up to global rotations, the dimension of the ground state manifold is \( D = F-K = q(n-1)-2n \). Note that for \( n=3 \), we get \( D=0 \) for the triangle (\( q=3 \)) and \( D=2 \) for the tetrahedron (\( q=4 \)).

Ground State Degeneracy

For a lattice built up of frustrated units the Hamiltonian can be written as:

\[ H = \frac{J}{2} \sum_{\alpha=1}^{N} L_\alpha^2 \]

where \( \alpha \) runs over all \( N \) units. If the lattice consists of corner-sharing units of \( q \) spins with \( n \) components we have:

\[ F = N(n - 1)q/2, \quad K = Nnq \]
\[ D = F - K = N[n(q - 2) - q]/2 \]

Here we have assumed that the constraints are independent. If \( n = 3 \), the dimension of the ground state manifold becomes extensive at \( q = 4 \), i.e., the pyrochlore antiferromagnet has an extensive ground state dimension. In general, \( D \) becomes extensive for \( n > 3 \), \( q > 4 \).

It turns out that the Kagome lattice also has an extensive \( D \) because of dependent constraints.
Energy barriers and order by disorder

The topology of the ground state is another important property. Is it possible to connect two ground states continuously or there are energy barriers separating them. When the temperature becomes finite, the state of equilibrium is the one that minimizes the free energy. Whereas the internal energy of every ground state is the same, the number of accessible low energy states (entropy) may be much higher for one of them. In this situation the system effectively spends all the time around this particular ground state. In general, this is an ordered state and its selection is known as order by disorder because it is stabilized by thermal fluctuations.

Phase diagram for n component spins arranged in corner-sharing units of q. Ordered states, provided they exist, are selected for $q \leq 4$ and $n \leq 3$ with the exception of $q=4$ and $n=3$ which is marginal.

Quantum fluctuations can also stabilize an ordered state out of the mechanism of order by disorder. In some cases, this ordered state has no classical analog. Let us consider the following simple example of a spin 1/2 Heisenberg model on a chain with nearest and next-nearest neighbor interactions.

For $J_1 = 2J_2$, this Hamiltonian can be expressed as:

$$H_{MG} = J_2 \sum_\alpha L_\alpha^2 = J_2 \sum_\alpha P^{L_\alpha=3/2} + C \quad \text{with} \quad L_\alpha = S_{\alpha 1} + S_{\alpha 2} + S_{\alpha 3},$$

where $\alpha$ denotes a triangular unit. The operator $P^{L_\alpha=3/2}$ projects the spins of the triangular plaquette $\alpha$ onto the $L_\alpha = 3/2$ subspace. This model is known as the Majumdar-Ghosh Hamiltonian.

Quantum Frustration

Therefore a state containing one single per triangular plaquette is a ground state. There are only two possible ground states:

![Diagram](image)

This ordered quantum state breaks spontaneously the $Z_2$ symmetry of $H_{MG}$ and it is known as valence bond solid or crystal. The elementary low energy excitations are deconfined spinons ($S=1/2$ excitations) that carry a topological defect called soliton:

![Diagram](image)
Confinement of fractional excitations in higher $D$

Valence bond crystal ground state of a 2D Hamiltonian (broken $Z_4$ symmetry):
Confinement in 2D
Klein generalized this idea to other lattices (D. J. Klein, J. Phys. A: Math. Gen. 15, 661 (1982). Let’s consider the following case for the square lattice:

\[ H = J_1 \sum_{\langle i,j \rangle} S_i \cdot S_j + J_2 \sum_{\langle\langle i,j \rangle\rangle} S_i \cdot S_j + K \sum_{\alpha} (P_{ij}P_{kl} + P_{il}P_{jk} + P_{ik}P_{jl}) \]

For \( J_2 = J_1/2 \) and \( K = J_1/8 \) we obtain:

\[ H_K = \frac{3J_1}{2} \sum_{\alpha} \mathcal{P}S_{\alpha=2} \]

Massive ground state degeneracy

\[ H_K = \frac{3J_1}{2} \sum_{\alpha} \mathcal{P} S_{\alpha=2} \]

\[ D \propto 2^L \]

Where \( L \) is a linear dimension of the system.
**Dimensional Reduction**

Local $Z_2$ order parameter on each diagonal zig-zag chain:

There is no effective coupling between order parameters on different chains!

![Diagram](image)

$D \propto 2^L$

Where $L$ is a linear dimension of the system.

The dimensionality is spontaneously reduced in the low energy sector of $H_K$!

Deconfined fractional excitations in 2D
Deconfined fractional excitations in 2D
Perturbations

One dimensional behavior and spinon excitations

\[ T \]

QPT

\[ g \]
Checkerboard Lattice

\[ H = J \sum_{\langle ij \rangle, \alpha} H_{ij}^\alpha + K \sum_\alpha (H_{ij}^\alpha H_{kl}^\alpha + H_{il}^\alpha H_{jk}^\alpha + H_{ik}^\alpha H_{jl}^\alpha), \]

where \( H_{ij}^\alpha = \vec{S}_i^\alpha \cdot \vec{S}_j^\alpha \). We can rewrite \( H \) as:

\[ H = \frac{J_1}{2} \sum \vec{S}_\square^2 + \frac{J_2}{4} \sum \vec{S}_\square^4, \]

For \( K = K_c = 4J/5 \), \( H \) becomes a Klein Hamiltonian:

\[ H_K = \frac{12}{5} J \sum \vec{P}_{S_T=2}^\square \]


We will assume that the dimer coverings, although non-orthogonal, they are linearly independent.
Deconfined Spinon Excitations
We denote the sites of the dual lattice $\mathcal{L}^D$ by $r$ and the link between the sites $r$ and $r+e_\mu$ by the pair $(r, \mu)$ where $e_\mu$ is a primitive vector of $\mathcal{L}^D$. We introduce the quantity $S_{r,\mu}^z$ on the links of the dual lattice (sites of the original lattice $\mathcal{L}$). The charge is $+1/2$ if the arrow flows from $\mathcal{A}$ to $\mathcal{B}$ and $-1/2$ if the arrow flows from $\mathcal{B}$ to $\mathcal{A}$.

The local zero divergence condition can now be written as:

$$\sum_{i\in\square} S_i^z = 0$$

Then, the operators:

$$\mathcal{U}_\alpha = e^{i\phi_\alpha} \sum_{i\in\alpha} S_i^z$$

are the local U(1) symmetry transformations of the theory under consideration.

We introduce now an oriented angular momentum variable on the links of $L^D$:

$$l_\mu(r) = \sigma_r(S^z_{r,\mu} + \frac{1}{2})$$

where $\sigma_r = 1(-1)$ if $r \in A$ ($B$).

By introducing the lattice gradient:

$$\nabla_\mu l_\nu(r) = l_\nu(r) - l_\nu(r - e_\mu)$$

We can rewrite the ice-rules as an explicit zero divergence condition on the vector field $l_\mu(r)$:

$$\sum_{\mu=1,2} \nabla_\mu l_\mu(r) = 0.$$
We now conjecture that the long range fluctuations of the coarse grained field $E$ is governed by the probability distribution:

$$p(E) \propto e^{-\frac{\kappa}{2} \int_V E^2}$$

Configurations that locally minimize the (coarse grained) field strength maximize the number of flippable plaquettes. The last equation implies that the gauge field is in a Coulomb phase. It is now straightforward to deduce the long distance correlator:

$$\langle E_\mu(0) E_\nu(r) \rangle = \frac{1}{4\pi K} \frac{[\delta_{\mu\nu} - 3\hat{r}_\mu \hat{r}_\nu]}{r^d}$$

The introduction of spinons (monomers) and defects violates the local zero divergence constraint. The corresponding charge is given by the divergence of $E$:

$$Q = \sum_{\mu=1,2} \nabla_\mu l_\mu(r)$$

- $Q = -2$
- $Q = 1$
The spinon-spinon interaction is determined by the change in the total entropy as a function of the distance between two monomers inserted in the lattice. The critical dipolar fluctuations of the underlying dimer field produce an effective Coulomb interaction which arises from the local conservation law of zero divergence:

\[ V(r) = \gamma q^2 \ln \frac{r}{a} \text{ for } \frac{r}{a} \ll 1 \]
Plasma of Spinons

$Q = -2$

$Q = 1$

$Q = 1$
This Rokhsar-Kivelson (RK) term is the minimal process that connects two different dimer coverings. The low energy theory in the presence of a perturbation that moves the system away from the Klein point is a dimer model on a non-orthogonal dimer basis.
The RK process cancels out exactly for the most natural perturbation: $K \neq K_c = 4J/5$. In this case, the next minimal processes that contributes with a non-zero matrix element are:
Thermally Driven Deconfinement

$T$ - Temperature

Dilute plasma of spinons

VBC I

VBC II

$K_c$ - Classical $T=0$ Critical Point
Summary

- Strongly frustrated systems have a very high low temperature entropy. There is an optimally frustrated case for which this entropy is accumulated at $T=0$, i.e., the ground state becomes massively degenerated.

- In this situations, thermal and quantum fluctuations play a fundamental role in determining the low energy physics of the system under consideration.

- Thermal fluctuations can select an ordered state out of the mechanism of “order by disorder”.

- Quantum fluctuations can even select a “quantum” ordered state with no classical analog like the valence bond crystal.

- The massive ground state degeneracy can also appear in quantum systems even when the quantum fluctuations are fully incorporated. These situations can lead to extremely interesting low energy phenomena like: dimensional reduction, fractional excitations in higher dimensions, an emergent electrodynamics for spinons and defects.

- Perturbations away from the optimally frustrated point are usually relevant for stabilizing some particular ordering at $T=0$. 